ANALYZING THE USE OF GENERALIZED HYPERBOLIC DISTRIBUTIONS TO VALUE AT RISK CALCULATIONS*

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RESUMO

O objetivo deste artigo é analisar o uso das distribuições Hiperbólicas Generalizadas (GH) para modelar a taxa de câmbio Dólar/Real de forma a obter medidas mais precisas de VaR (Value at Risk). Depois da estimação dos parâmetros da GH, várias distâncias foram calculadas para verificar a qualidade do ajuste da distribuição Normal e da família de distribuições GH aos dados empíricos. As distribuições GH têm mostrado serem mais adequadas para a modelagem da taxa de câmbio Dólar/Real, já que elas produzem distâncias menores, especialmente nas caudas. Adicionalmente, várias metodologias de cálculo do VaR foram comparadas usando o teste de Kupiec: Simulação Histórica, RiskMetrics®, Normal incondicional, GH, Normal Inversa Gaussiana (NIG) e Hiperbólica, bem como modelos GARCH usando Normal, GH, Hiperbólica e NIG. A distribuição GH e suas subclasses mostraram melhores resultados do que a Normal incondicional. O uso de um modelo GARCH para fazer previsões da volatilidade mostrou também resultados satisfatórios. Dois métodos de estimação foram usados: Máxima Verossimilhança e Minimização de FOF; porém, ambos produziram resultados similares. Dado que a Máxima Verossimilhança se mostrou ser a mais rápida, recomenda-se este método. Finalmente, recomenda-se o uso de uma família de distribuições GH reescalada por uma volatilidade GARCH e estimada por Máxima Verossimilhança.

Palavras-chave: valor em risco, distribuições hiperbólicas generalizadas, backtesting.

ABSTRACT

The goal of this paper is to analyze the use of the Generalized Hyperbolic (GH) Distributions to model the US Dollar/Brazilian Real exchange rate in a way to produce more accurate VaR (Value at Risk) measurements. After the GH parameters estimation, several distances were calculated to verify the fitting quality of Normal distribution and GH distribution family to empirical data. The GH Distributions had shown to be more adequate for modeling the US Dollar/Brazilian Real exchange rate, since they produced smaller distances, especially in tails. Additionally, several methodologies for VaR calculation were compared using the Kupiec test: Historical Simulation, RiskMetrics®, unconditional Normal, GH, Normal Inverse Gaussian (NIG) and Hyperbolic, and GARCH models using Normal, GH, Hyperbolic and NIG. The GH Distribution and its subclasses showed better results than unconditional Normal. The use of a GARCH model for volatility forecasting produced satisfactory results, being the main factor of success. Two estimation methods were used: Maximum Log-Likelihood and Minimization of the FOF distance; but both produced similar results. As the Maximum Log-Likelihood showed to be faster we recommend this method. Overall, our recommendation the use of a GH family distribution re-scaled by a GARCH volatility and estimated by Maximum Log-Likelihood.

Key words: value at risk, generalized hyperbolic distributions, backtesting.

JEL classification: C13, C15, C16.

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1 Introduction

The importance of the risk management is increasing in the recent years, and became one of the major concerns of financial and non-financial institutions. The risk management can be divided is several areas: market, credit, liquidity and operational. This paper will focus on the Market Risk, that is specially important in emerging countries, where the volatility of the markets are extremely high.

This paper aims to model the US Dollar/Brazilian Real exchange rate using the Generalized Hyperbolic (GH) Distribution, with focus on risk measurement. This distribution has been shown more adequate than the Normal Distribution to model financial assets (see, for example, Eberlein e Prause, 2000) because of two characteristics: fat tails and asymmetry. The purpose of this paper is to use GH distribution to generate more accurate VaR (Value at Risk) measurements to this exchange rate than those that use a Normal distribution.

The parameters estimation of the GH to the Dollar/ Real exchange rate data will be done by maximum log-likelihood and also minimizing the distance proposed by Farias, Ornelas and Fajardo (2002), that we will refer as FOF distance. This distance, similar to the Anderson-Darling distance, gives special importance to the tails of the distribution, being more adequate if the objective is VaR calculation.

After the parameters estimation, several kinds of distances will be calculated between empirical and theoretical estimated distributions. Then, significance tests will verify the hypothesis that empirical distribution is equal to theoretical distribution. Finally, the VaR will be calculated for the Dollar/Real exchange rate in the considered period, and also a backtest will be performed to verify models effectiveness.

The paper is structured as follows: on the next section, a short overview of GH will be presented; on section three, the data used will be described; on section four the parameters estimation will be done together with a description of the several distances used and significance tests; on section five, VaR will be calculated along with Backtest, using several methodologies; and section six concludes the paper.

2 Value at Risk and Generalized Hyperbolic Distributions

Several methodologies can be used to estimate the market risk measures. The choice will depend on several aspects, such as the kind of portfolio and resources available to implement the risk management system. The main three methodologies are: Parametric, Historical Simulation and Monte-Carlo Simulation. This paper focuses on the parametric approach.

The Parametric approach assumes that the returns follow a certain probability distribution, and then the parameters of the distribution (for example, the volatility) are estimated. Using the probability distribution function and the parameters, the risk measures can be calculated. The most common and used risk measure is the Value at Risk (VaR) that is based on a quantile of the distribution:

$$P[R < -VaR(\alpha)] = 1 - \alpha \tag{2.1}$$

where R are the returns and

α is the significance level with which the VaR is being calculated.

Usually, the estimation of parametric VaR assumes that financial assets follow a Normal distribution. Unfortunately, this is not the most adequate distribution to model financial assets. Rydberg (1997) enumerates several stylized facts from financial data that a distribution must show. Among them are the fat tails and negative asymmetry.

The fat tails are related to the fact that real world distributions, in general, have fatter tails than the Normal distribution. This means that the probability of exaggerated returns, positive or negative, are more common in real world data than in Normal distributions.

Another stylized fact is that real world distributions, in special shares, have a slight negative asymmetry. A possible explanation for this fact is that agents react more vigorously to negative information than to positive information.

Barndorff-Nielsen (1977) presented a model to represent the distribution of sand particles size using the so-called Generalized Hyperbolic distributions (GH). Eberlein and Keller (1995) introduced the Hyperbolic distributions (a subclass of the GH) in finance, to try to represent stylized facts that the Normal and other distributions could not represent using German data.

The following density defines the Generalized Hyperbolic:

$$GH(x;\lambda,\alpha,\beta,\delta,\mu) = a(\lambda,\alpha,\beta,\delta)(\delta^2+(x-\mu)^2)^{(\lambda-1/2)/2}\times K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right) e^{(\beta(x-\mu))} \quad (2.2)$$

where:

K_x is the modified Bessel function and

$$a(\lambda,\alpha,\beta,\delta) = \frac{\left(\alpha^2-\beta^2\right)^{\lambda/2}}{\sqrt{2\pi}\alpha^{(\lambda-0,5)}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})}$$

The parameters are real numbers with the following restrictions (see Prause, 1999):

$$\begin{array}{ll} \delta \geq 0,\ |\beta| < \alpha & \text{if } \lambda > 0 \\ \delta > 0,\ |\beta| < \alpha & \text{if } \lambda = 0 \\ \delta > 0,\ |\beta| \leq \alpha & \text{if } \lambda < 0 \end{array}$$

The parameter δ is a scale factor, compared to the σ of a Normal distribution by Eberlein (2000), and μ is a location parameter. The parameters α and β determine the distribution shape and λ defines the subclasses of GH and is directly related to tail fatness (Barndorf-Nielsen and Blæsild, 1981). The function $a(.)$ is introduced to guarantee that the cumulative density totals one. There are other parametrizations created to obtain a scale- and location- invariance. Appendix I describes three of them. These parametrizations are important when one needs to change a parameter, while maintaining the shape of the distribution.

The GH has several subclasses, among them the Hyperbolic and the Normal Inverse Gaussian (NIG). Setting λ = -1/2, we get the NIG, and with λ = 1 we get the Hyperbolic distribution. Gaussian is a limiting distribution of GH, when $\delta \rightarrow \infty \rightarrow \alpha$ and $\delta/\alpha \rightarrow \sigma^2$

The GH distributions and its subclasses have been used to calculate VaR by several papers. Bauer (2000) used a symmetric Hyperbolic distribution to perform VaR calculations, and used data from German stocks and international stock indexes (DAX, Dow Jones and Nikkei) from 1987 to 1997. His results showed that the model with Hyperbolic distribution outperformed a model with a Normal distribution with a volatility given by an Exponential Weighting Moving Average (EWMA).

Eberlein, Keller and Prause (1998) also use the Hyperbolic distribution for VaR calculations of German Stocks. They found that the loss function derived from the hyperbolic model is in accordance with the empirically observed one. Another study, Eberlein and Prause (2000) used several types of Generalized Hyperbolic distribution to linear portfolios of German stocks. Results were compared with the Historical Simulation, Multivariate Normal and a Normal distribution with an I-GARCH model to the volatility. Also, Fajardo and Farias (2004) uses the Generalized Hyperbolic and two subclasses (NIG and Hyperbolic) distribution to calculate the VaR of Brazilian Stocks, finding that the Normal distribution under or over estimate the VaR depending on the confidence level, with the GH and its subclasses having a better fit.

Although GH does a better job than Normal distribution, it is worth nothing that the only GH subclass that is closed under convolutions is the NIG, and therefore if we choose any other subclass we will demand more computational effort to estimate the distribution of a n-day return or a return of a portfolio, since the sum of two GH distributions is not necessarily another GH. Therefore, the Normal distribution and Historical simulation approaches do a simpler job, demanding less computational effort.

For the n-day return, it is suggested to multiply the characteristic functions in order to obtain the characteristic function of the n-day return (under the assumption of independent increments), and then we must apply the inverse Fourier transformation to recover the n-day distribution. For the return of a portfolio using GH distributions, Prause (1999) proposes several approaches. We suggest the approach where the shape of the GH is estimated using a longer time period (more than one year) and then an up-to-date covariance matrix is used. Therefore, we have to choose a subclass of the GH, and a long-term estimate of the parameters. The use of a long-term shape parameter incorporates a high possibility of extreme events, even if no crash in the preceding 252 trading days have occurred.

3 Data description

The data used on this paper consists of US Dollar/Brazilian Real exchange rate from Reuters. The quote considered for calculations was simple average of bid and ask daily close, from 1/13/1999 to 08/29/2002. The initial date was chosen because it was free float exchange rate regime beginning in Brazil. Before that, there was a band system, where the Central Bank used to do buy and sell auctions to keep exchange rate inside the band. The behavior can be seen in Figure 1. So, in this paper, we considered only the free float regime.

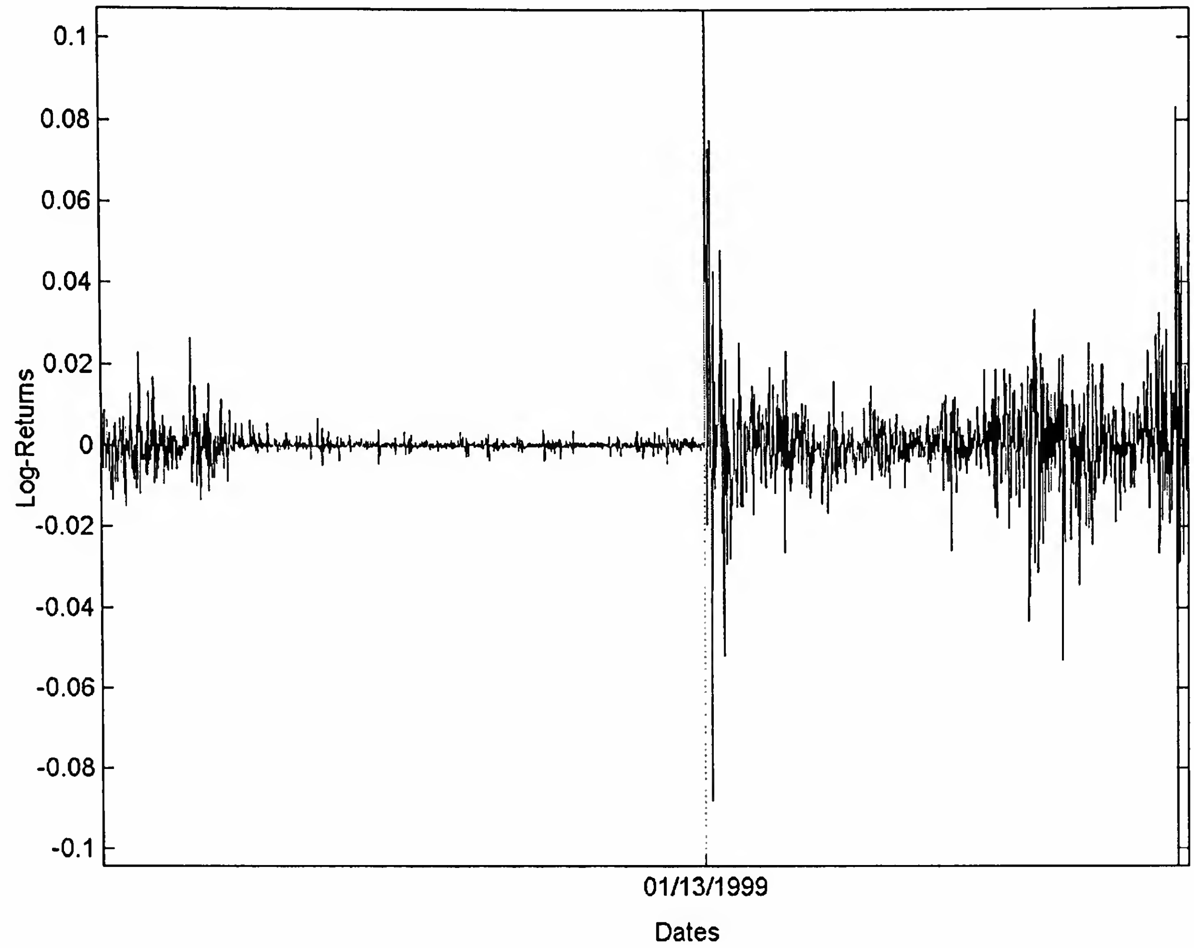
Returns used were logarithmic, according to the following formula:[1]

$$R_t = Ln\left[\frac{(Bid_t + Ask_t)}{(Bid_{t-1} + Ask_{t-1})}\right] \tag{3.1}$$

1 The use of the bid and ask instead of the last trade price is due to the fact that sometimes the last trade is done some minutes before the end of trading. Then, we may have a last trade price higher than the ask close price or lower than the bid close price, which makes no sense.

Where R_t is the return from day *t*, Bid_t is close bid price from day *t*, Ask_t is close ask price from day *t*, and *Ln* is neperian logarithmic.

Figure 1 – Return of US dollar/Brazilian real exchange rate



In Table 1 we have main information on the sample.

Table 1 – Sample return data

Sample Data	
Return Average	0.00100
Return Standard Deviation	0.01310
Asymmetry	0.23560
Kurtosis	21.2117
Maximum Return	0.1075
Minimum Return	-0.1041
Number of Observations	945

4 Parameter estimation and distance analysis

A frequent problem in statistic and finance is to measure the goodness-of-fit of a theoretical distribution to real world data. To measure how close, or how far, is an empirical distribution from a theoretical distribution, several distances have been proposed. Among them, we can cite three: the Kolmogorov distance; Kuiper distance and the Anderson-Darling distance.

The Kolmogorov distance (see, for example, Massey, 1951) is defined as the greatest distance between empirical distribution and theoretical distribution, for all possible values:

$$D_{kol} = \max_{x \in \Re} \left| f_{Emp}(x) - f_{Teo}(x) \right| \tag{4.1}$$

where f_{Emp} is the empirical cumulative density function and f_{Theo} is the continuous and completely specified theoretical cumulative density function.

f_{Emp} can be defined by:

$f_{Emp}(x) = (number\ of\ X_i 's \leq x)/n$

where $X_i 's$ are the sample's elements and n is the sample number of elements.

The Kuiper distance (see Kuiper, 1962) is similar to the Kolmogorov distance, but it considers the deviation direction, adding the greatest distances upwards and downwards:

$$D_{Kui} = \max_{x \in \Re} \left\{ f_{Emp}(x) - f_{Theo}(x) \right\} + \max_{x \in \Re} \left\{ f_{Theo}(x) - f_{Emp}(x) \right\} \tag{4.2}$$

The Anderson and Darling (1952) paper proposes a distance that would be viewed as Kolmogorov distance with weight. Weighting can be defined giving special importance to distribution tails, and so being especially relevant to VaR calculations. The formula of this distance with tail emphasis is:

$$D_{ad} = \max_{x \in \Re} \frac{\left| f_{Emp}(x) - f_{Teo}(x) \right|}{\sqrt{f_{Teo}(x)\left(1 - f_{Teo}(x)\right)}} \tag{4.3}$$

The AD distance is especially interesting to perform VaR calculations, since it is more sensitive in tails than in distribution's middle range.

Another interesting distance to VaR calculations is the FOF distance (see Farias, Ornelas and Fajardo, 2002). It uses AD distance weight function and worries with deviation's direction like Kuiper distance. So, it captures Kuiper and AD distances strengths. The FOF distance is the following:

$$D_{FOF} = \max_{x \in \Re} \frac{f_{Emp}(x) - f_{Teo}(x)}{\sqrt{f_{Teo}(x)\left(1 - f_{Teo}(x)\right)}} + \max_{x \in \Re} \frac{f_{Teo}(x) - f_{Emp}(x)}{\sqrt{f_{Teo}(x)\left(1 - f_{Teo}(x)\right)}} \tag{4.4}$$

For the distances mentioned previously we can verify data fitting quality through statistical significance tests. To do this we test the null hypothesis that empirical distribution is equal to theoretical distribution. For the Kolmogorov and Kuiper distances, there is a formula to obtain the critical values, but for the other distances it is necessary to use Monte Carlo Simulation to obtain the critical values.

Another important issue is GH parameters estimation. Blæsild and Sørensen (1992) use maximum log-likelihood estimation to Hyperbolic distributions, and Fajardo e Farias (2004) also uses log-likelihood, but to estimate the general case of the GH. Prause (1999) proposed several estimation methods through minimization of the Anderson-Darling distance and minimization of certain percentiles. In this paper, we estimate the parameters through maximum log-likelihood and also through FOF distance minimization.

Regarding minimum FOF estimation, distribution tails are focused, leaving less importance to middle range. This is reasonable if we want estimation for VaR calculations.

We developed a program that uses maximum log-likelihood to estimate the GHD parameters, because Prause (1999) showed that this estimation method is the only one non-biased, among a large class of methods used in the literature. This method was also used by Blæsild and Sørensen (1992) in the Hyp software, in order to estimate only the multivariate hyperbolic subclass ($\lambda = 1$) parameters. The maximum log-likelihood parameters are those that maximize the following likelihood function:

$$L = \sum_{i=1}^{n} \log(gh(x_i;\alpha,\beta,\delta,\mu,\lambda)) \tag{4.5}$$

This estimation consists in a numerical optimization procedure. We use the Downhill Simplex Method which makes no use of derivatives, developed by Nelder and Mead (1965), with some modifications (due to parameter restrictions). It is worth noting that Prause (1999) used a Bracketing Method. The Downhill Simplex Method requires starting values to begin optimization, in this case we followed Prause (1999) who used a symmetric distribution ($\beta = 0$) with a reasonable kurtosis ($\xi \approx 0.7$) to equalize the mean and variance of the GHD to those of the empirical probability distribution. This is done because when we use a symmetric distribution and fix the kurtosis, we reduce the computational effort.

In all numerical optimizations we have to define the tolerance of the search, and we decided to use 1×10^{-10} This tolerance was applied in absolute ways to the function evaluation and to the parameters sum variation. The numerical maximum likelihood estimation does not have a convergence analytical proof, but even using different starting values it has showed empirical convergence. (Prause, 1999). The Minimum FOF estimation was implemented in a similar way. The algorithms were implemented using MatLab® software, to GH, NIG and Hyperbolic, and generates the following parameters:

Table 2 – Parameters estimated

Distribution	Estimation Method	α	β	δ	μ	λ
GH	Min FOF	34.185	-0.001774	0.001752	0.001502	0.000
GH	Max LogLikelih.	20.412	0.150185	0.006388	0.0006121	-0.727
NIG	Min FOF	15.638	-0.000729	0.005390	0.0012116	-0.500
NIG	Max LogLikelih.	32.770	3.413905	0.005270	0.0004294	-0.500
Hyperbolic	Min FOF	71.633	-0.294398	2.35E-307	-3.76E-05	1.000
Hyperbolic	Max LogLikelih.	132.448	0.121929	7.55E-09	3.85E-11	1.000

Due to overparametrization, estimation of GH distribution by minimum FOF generated several local optimums (that didn't happen with Hyperbolic and NIG). In order to avoid backtesting problems, we presented the global optimum parameter estimation but didn't compare the results for Value at Risk purposes.

After the parameters estimation, the distances were calculated together with the hypothesis tests, as presented in Table 3:

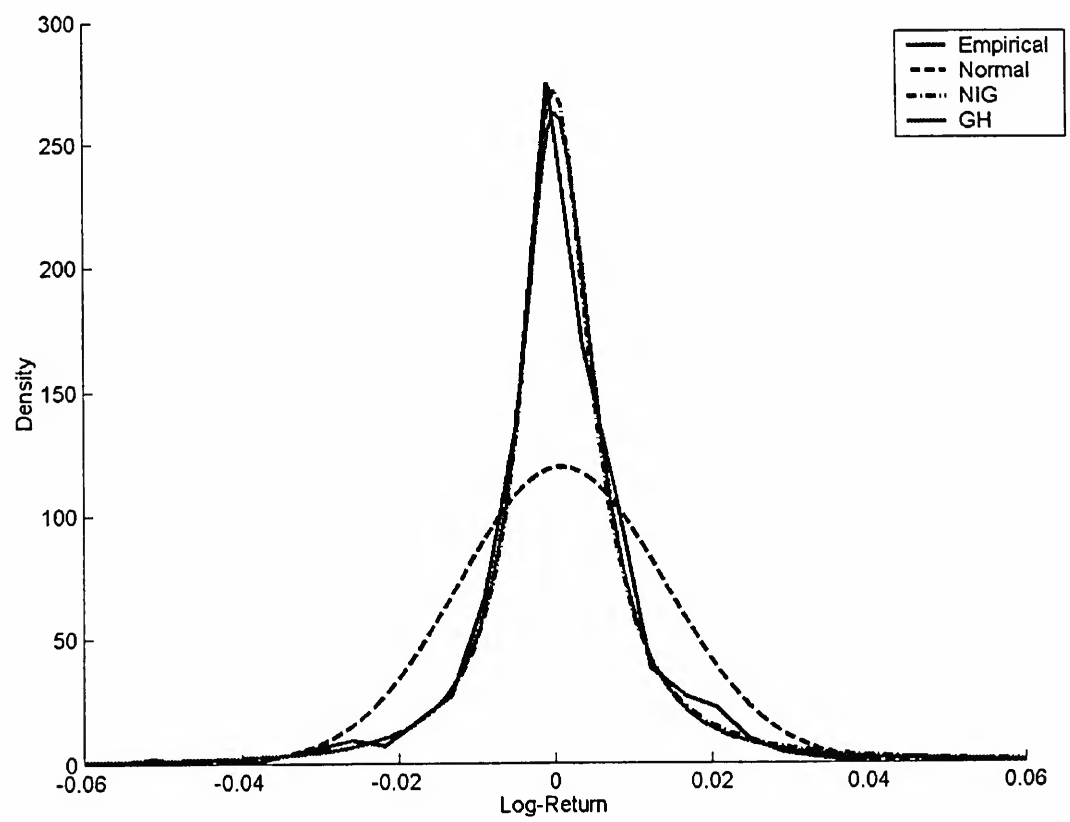
Table 3 – Distances and significance tests

Distribution	Estimation Method	Kolmogorov Distance	P-value KS	Kuiper Distance	P-value Kuiper	AD Distance	FOF Distance
Normal		0,133400	0	0,26200	0,00E+00	100429,91	103564,37
GH	Min FOF	0,037902	0,12949	0,07048	2,72E-03	0,09138	0,11655
GH	MaxLogL	0,031184	0,31242	0,06157	1,93E-02	0,06411	0,12487
NIG	Min FOF	0,036433	0,15941	0,05831	3,63E-02	0,07369	0,11744
NIG	MaxLogL	0,029895	0,36216	0,05859	3,44E-02	0,08168	0,14304
Hyperbolic	Min FOF	0,122651	7,14E-13	0,19270	4,43E-29	0,27241	0,44982
Hyperbolic	MaxLogL	0,073149	7,48E-05	0,08453	6,22E-05	16,84139	17,72214

As can be seen in Table 3, the Normal distribution always has higher distances than the GH and its subclasses. Among GH family, the worst performance is from Hyperbolic. GH and NIG are the distributions that are closest to empirical distribution.

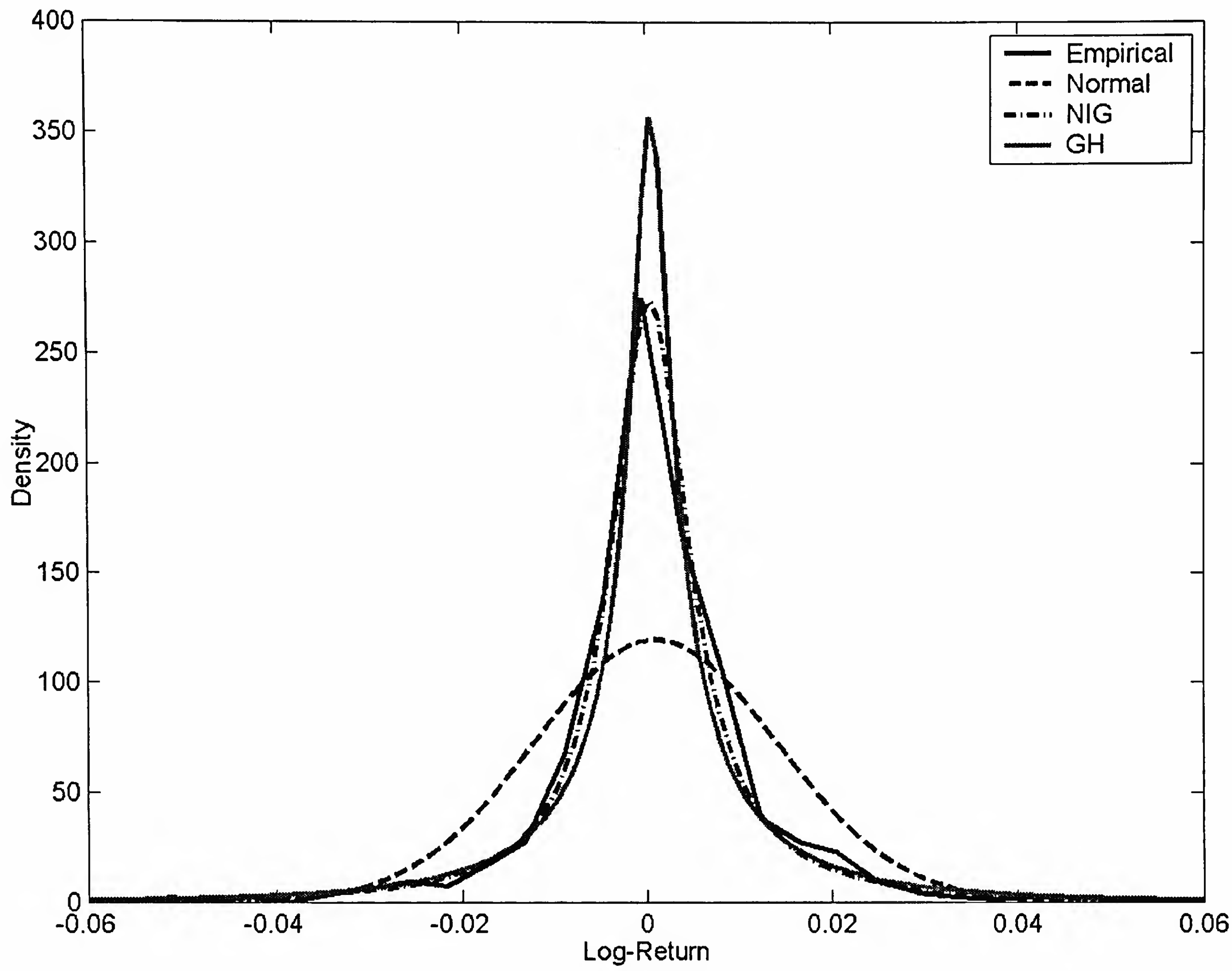
We can also analyze the goodness-of-fit of distributions to real data through a visual inspection of graphs. Figure 2 shows the GH distributions estimated by maximum log-likelihood, with Normal and Empirical distribution, and Figure 3 shows GH distributions estimated by minimum FOF, together with Normal and Empirical distribution.

Figure 2 – Density x log-return – maximum loglikelihood estimation



As we can see on Figure 2, the Normal distribution presents a density, in middle range, lower than empirical. But NIG and GH are very close to the empirical, and also close to each other (the optimal λ from the GH is -0.7 close to the -0.5 from NIG). On Figure 3, the distribution that seems to best approximate the empirical is the NIG.

Figure 3 – Density x log-return – minimum fof estimation



5 VaR CALCULATION

To verify the applicability of GH distributions to VaR calculations, a backtest was performed using several methodologies described later. To each methodology, parameters were estimated to the first 252 days from sample. After that VaR for Dollar active positions was calculated and a comparison with actual return was made. The period of 252 days was then rolled one day until the last 252 days of the sample were reached. Then, it was counted how many times effective return was outside VaR limits. With this data, the Kupiec (1995) test (see Appendix B for details), with the null hypothesis that the predicted and actual outliers are equal, was applied to verify which methodologies calculated more effectively the VaR.

The methodologies used were the following:

- Historical Simulation: based on a moving window of the previous 252 days of actual returns;

- Riskmetrics®: uses a Normal distribution and a EWMA with decay factor equal to 0.94 to calculate variance;
- Normal distribution with GARCH(1,1) volatility: uses a Normal distribution and a GARCH model to forecast the volatility;
- Unconditional Normal – uses a Normal distribution with mean and standard deviation calculated based on data from the previous 252 days;
- Unconditional GH, NIG and Hyperbolic estimated by Maximum Log-likelihood – uses a GH family distribution with parameters calculated by Maximum Log-likelihood using data from the previous 252 days;
- Unconditional NIG and Hyperbolic estimated by Minimum FOF – uses a GH family distribution with parameters calculated by Minimum FOF using data from the previous 252 days;
- GH, NIG and Hyperbolic estimated by Maximum Log-likelihood with a GARCH(1,1) volatility – first, a GH family distribution with parameters calculated by Maximum Log-likelihood using data from the previous 252 days. Then, the a GARCH(1,1) model is used to provide a forecast of the volatility. Finally, the fitted distribution is re-scaled using the volatility given by the GARCH model;
- NIG and Hyperbolic estimated by Minimum FOF with a GARCH(1,1) volatility – first, a GH family distribution with parameters calculated by Minimum FOF using data from the previous 252 days. Then, the a GARCH(1,1) model is used to provide a forecast of the volatility. Finally, the fitted distribution is re-scaled using the volatility given by the GARCH model.

The results are on Table 4.

As we can see on Table 4, all methodologies passed the Kupiec test to 5% and 10% VaR with 99% confidence level to the null hypothesis. But to 1% VaR only the GARCH models, Historical Simulation and unconditional NIGs and GH passed the test with confidence of 99%. We would say that the volatility given by a GARCH model is the most relevant factor for the success of the model.

But the distribution used is also relevant. Although the Normal with GARCH model has passed on the Kupiec test, it seems that the number of outliers given by this model is underestimated (0.87%, 3.17% and 7.36%, where the expected is 1%, 5% and 10%). The GH distributions and its subclasses with GARCH model gave the best results overall (in terms of outlier probability), no matter the estimation procedure. Although the unconditional hyperbolic distribution was rejected by the Kupiec of the 1%-VaR, the conditional Hyperbolic using GARCH had one of the best performances. Nevertheless, we could not point out specifically which is the best GH subclass - a larger time series would be necessary for that.

Regarding the estimation method, overall results are similar. As the Maximum Log-Likelihood was faster in our implementation, we recommend this method, instead of minimizing the FOF distance.

Table 4 – Backtest results

	Target VaR Probability	1%	5%	10%
	Outliers Number	21	44	63
Normal	Outliers Probability	3,03%	6,35%	9,09%
	P-value Kupiec	1,51E-05	0,1171	0,4186
	Outliers Number	12	36	58
Historical	Outliers Probability	1,73%	5,20%	8,37%
Simulation	P-value Kupiec	0,0795	0,8151	0,1421
	Outliers Number	20	41	61
RiskMetrics	Outliers Probability	2,89%	5,92%	8,80%
	P-value Kupiec	4,85E-05	0,2815	0,2843
	Outliers Number	6	22	51
GARCH	Outliers Probability	0,87%	3,17%	7,36%
	P-value Kupiec	0,7163	0,0184	0,0155
	Outliers Number	13	45	74
GH	Outliers Probability	1,88%	6,49%	10,68%
(Max LogLik.)	P-value Kupiec	0,0380	0,084	0,5556
	Outliers Number	16	46	71
Hyperbolic	Outliers Probability	2,31%	6,64%	10,25%
(Max LogLik.)	P-value Kupiec	0,0031	0,059	0,8302
	Outliers Number	13	46	73
NIG	Outliers Probability	1,88%	6,64%	10,53%
(Max LogLik.)	P-value Kupiec	0,0380	0,0590	0,6420
	Outliers Number	15	45	70
Hyperbolic	Outliers Probability	2,16%	6,49%	10,10%
(Min FOF)	P-value Kupiec	0,0079	0,089	0,9597
	Outliers Number	11	44	72
NIG	Outliers Probability	1,59%	6,35%	10,39%
(Min FOF)	P-value Kupiec	0,1559	0,1236	0,7629
GARCH	Outliers Number	6	38	78
GH	Outliers Probability	0,87%	5,48%	11,26%
(Max LogLik.)	P-value Kupiec	0,7163	0,5651	0,2792
GARCH	Outliers Number	6	32	66
Hyperbolic	Outliers Probability	0,87%	4,62%	9,52%
(Max LogLik.)	P-value Kupiec	0,7163	0,6400	0,6739
GARCH	Outliers Number	6	37	75
NIG	Outliers Probability	0,87%	5,34%	10,82%
(Max LogLik.)	P-value Kupiec	0,7163	0,6852	0,4757
GARCH	Outliers Number	6	35	75
Hyperbolic	Outliers Probability	0,87%	5,05%	10,82%
(Min FOF)	P-value Kupiec	0,7163	0,9514	0,4757
GARCH	Outliers Number	6	46	89
NIG	Outliers Probability	0,87%	6,64%	12,84%
(Min FOF)	P-value Kupiec	0,7163	0,0590	0,0164

6 Conclusion

The unconditional GH distributions showed to be more adequate than Normal for the US Dollar/Brazilian Real exchange rate modeling. That is evidenced by a distance analysis between empirical and theoretical distributions. On the backtesting, all methods were approved by the Kupiec test for the 5% and 10% VaR. For the 1%-VaR, recommended by the Basel Committee, the Historical Simulation, unconditional NIG and GH, the all conditional distributions using GARCH were approved by Kupiec at 1% significance level. If we consider Kupiec at 5% confidence level, only the GARCH models and Historical simulation are approved. Therefore, we recommend the use of a GH subclass distribution with a volatility given by GARCH model, when calculating the 1% VaR.

Although it is the most used test, the Kupiec test is known as being a not very powerful test (e.g. it fails to reject the null hypothesis when it is actually false). Therefore, a suggestion for further research would be to implement other backtesting procedures such as the one proposed by Christorfessen (1998), that considers also the independence of the outliers, beside the proportion.

Another suggestion for further research is to apply a GH family distribution with a GARCH process for other kinds of Brazilian assets, such as bonds, and also to portfolios. The application for portfolios is special relevant, as it simulates real world applications of Risk Managers.

Finally, the same methodology may be extended to calculate other Risk Measures, such as the Expected Shortfall, since the VaR measure do not consider the size of tail losses.

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APPENDIX A – PARAMETERIZATIONS OF THE GENERALIZED HYPERBOLIC DISTRIBUTION

There are in the literature several parameterizations for the GH that is scale- and location-invariants. Three of them are presented to follow:

Second parametrization : $\zeta = \delta\sqrt{\alpha^2 - \beta^2}, \psi = \beta / \alpha$

Third parametrization : $\xi = (1+\zeta)^{-1/2}, \chi = \xi\psi$

Fourth parametrization $\overline{\alpha} = \alpha\delta, \overline{\beta} = \beta\delta$

Further details and a proof of the invariance of these parameterizations can be viewed at lemma 1.5 of Prause (1999).

APPENDIX B – THE KUPIEC TEST

The Kupiec (1995) test is the most used Backtesting for Value at Risk calculations. It is a test of the proportion of losses that exceeds VaR. The idea is to test whether the observed frequency of losses greater than VaR is consistent with the frequency predicted by the model. The null hypothesis is that the actual and predicted proportions of observations that exceed VaR are equal, i.e., the model is adequate. Under the null hypothesis, the number of losses beyond VaR follows a binomial distribution:

$$P(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

where n is the sample size and p is the expected probability, in this case the VaR significance level.

The test statistic is given by:

$$LR = 2\left(\ln\left((x/n)^x (1-(x/n))^{n-x}\right) - \ln\left(p^x (1-p)^{n-x}\right)\right)$$

where x is the number of VaR exceptions.

LR follows a chi-squared distribution with one degree of freedom.

Although the Kupiec test is the most commonly used, it has been criticized for two reasons. First, it is reliable only with very large samples (Dowd, 2002). Second, it odes not verify for the independence of the exceptions, so one period of high number of exceptions may counterbalance a period of low number, with the final result being close to the average number of exceptions.