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The Role of Identities in Jordan Algebras[1]

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Abstract: Born of quantum mechanics, but abandoned at birth by physicists, Jordan algebras recovered to lead a productive life in a variety of mathematical fields. Zel'manov's amazing classification of simple Jordan algebras of arbitrary dimension focused attention on the identities (identical relations satisfied by all elements of an algebra) – if clothes make the man, then identities make the algebra. We give a survey of the history of Jordan structure theory, stressing the roles of the s-identities (which separate special algebras, those with an associative parentage, from algebras of Albert type), the Clifford identities (which separate Jordan algebras of Clifford type with just two idempotent genes from the other special algebras), and Zel'manov's dread penteater identity (which heartily eats pentads and spits out the heart of a Jordan algebra of hermitian type).

1 The Jordan Program

Jordan algebras were conceived during physicist Pascual Jordan's search for an "exceptional" setting for quantum mechanics. In the usual interpretation of quantum mechanics (the "Copenhagen model"), the physical observables are represented by self-adjoint operators on Hilbert space (or Hermitian matrices). The basic operations on operators or matrices are multiplication by a complex scalar, addition, composition of operators (or multiplication of matrices), and forming the adjoint operator (or complex conjugate transpose matrix). But these underlying operations are not "observable": the scalar multiple of a hermitian object is not again hermitian unless the scalar is real, the product is not hermitian unless the factors happen to commute (are "simultaneously observable").

In 1932 Jordan proposed a program to *discover a new algebraic setting for quantum mechanics*, which would be freed from dependence on an invisible all-determining metaphysical matrix structure, yet would enjoy all the same algebraic benefits as the matrix model:

• *To study the (observable) intrinsic algebraic properties of hermitian matrices, without reference to the (unobservable) underlying matrix algebra*;

• *To capture the algebraic essence of the physical situation in formal algebraic properties that seemed essential and physically significant*;

• *To classify the abstract systems satisfying these formal axioms and discover new (non-matrix) systems.*

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2 The Jordan Operations

The first step in analyzing the algebraic properties of hermitian matrices or operators was to decide what the basic *observable operations* were. The most natural observable operation was forming polynomials, built up from the operations of multiplication by a real scalar, addition, and raising to a power. If an operator x takes a value a in a particular state, the operator $p(x)$ takes the value $p(a)$, raising no question of "simultaneous observability". By linearizing the quadratic squaring operation we obtain a symmetric bilinear operation

$$x \bullet y := \tfrac{1}{2}(xy + yx)$$

(now called the *Jordan product*, or simply the *bullet*). After some empirical experimentation, Jordan decided that everything could all be expressed in terms of this product.

3 The Jordan Axioms

The next step in the empirical investigation of the algebraic properties enjoyed by the model was to decide what crucial formal *identities* or *laws* the operations on hermitian matrices obey. The operation of composing polynomials leads to power-associativity $x^n \bullet x^m = x^{n+m}$, and Jordan discovered he could derive this from commutativity and a single identity of degree four (now called the *Jordan identity*)

$$x \bullet y = y \bullet x, \qquad x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x.$$

The outcome of all this experimentation was a distillation of the algebraic essence of quantum mechanics into an axiomatically defined algebraic system:.

Linear Jordan Definition. *A* **linear Jordan algebra** *over a ring of scalars* Φ *containing* $\frac{1}{2}$ *consists of a* Φ*-module equipped with a bilinear product* $x \bullet y$ *satisfying the commutative law and the Jordan identity.*

4 Quadratic Jordan Algebras

In order to handle Jordan rings, or algebras of characteristic 2, where $\frac{1}{2}$ no longer exists, the concept of linear algebra must be enriched to that of a quadratic algebra. In any linear Jordan algebra we can introduce an important *U-operator* and *Jordan triple product*, with derived V-operators, by

$$U_x := 2L_x^2 - L_{x^2},\ U_{x,z} := U_{x+z} - U_x - U_z,$$
$$V_{x,y}(z) := U_{x,z}(y) =: \{x,y,z\},\ V_x := 2L_x,\ V_x(z) = 2x \bullet z =: \{x,z\}.$$

In your (special) heart, you should always think of the U operator as "outer multiplication" (simultaneous left and right multiplication), and the V operators as left plus right multiplications:

$$U_x(y) \approx xyx, \quad \{x,y,z\} \approx xyz + zyx, \quad \{x,y\} \approx xy + yx.$$

We will frequently write the quadratic product simply as U_xy (omitting parentheses and using mere juxtaposition), except when y is a complicated expression, or we want to focus on U_x as operator.

Quadratic Jordan Definition. *A* **unital quadratic Jordan algebra** *consists of a Φ-module, a choice of* unit element 1, *and a product U_xy which is linear in the variable y and quadratic in x and strictly satisfies the following operator identities:*

$$U_1 = Id, \quad V_{x,y}U_x = U_xV_{y,x}, \quad U_{U_xy} = U_xU_yU_x$$

corresponding to $1z1 = z$, $xy(xzx) + (xzx)yx = x(yxz + zxy)x$, $(xyx)z(xyx) = x\big(y(xzx)y\big)x$ *in associative algebras.*

Unital algebras get a squaring operation $x^2 := U_x1$ free of charge from their unit. Non-unital algebras must have, in place of a unit, a quadratic squaring operation satisfying additional axioms; they are precisely all subspaces of unital algebras closed under U_xy and x^2.

Here *strictness* means that the identities continue to hold in all scalar extensions, equivalently, that all their linearizations hold in the original algebra. (This holds automatically if $\frac{1}{6} \in \Phi$, or if Φ is a field with at least 4 elements.) If $\frac{1}{2} \in \Phi$ then linear and quadratic Jordan algebras are categorically equivalent: we can translate back and forth between U and the bullet via $U_xy = 2x \bullet (x \bullet y) - x^2 \bullet y$, $x \bullet y = \frac{1}{2}\{x,y\}$.

The quadratic viewpoint leads naturally to the notion of a one-sided ideal. In a commutative linear algebra, there is no distinction between two-sided and one-sided ideals. But the quadratic product xyx has an *inside* and an *outside*; an *inner ideal* is a subspace $B \subseteq J$ closed under *inner multiplication* by the entire algebra, $U_BJ \subseteq B$. For example, in an associative matrix algebra under the Jordan operations, any left or right ideal is an inner ideal, and any element x determines a *principal* inner ideal U_xJ. The quadratic viewpoint also reveals the main culprits in the structure theory, the *trivial* elements z with trivial U-operator $U_z = 0$ ("two-sided" or "outer" zero divisors in the sense that $zxz = 0$ for all $x \in J$). The standard of decorum for Jordan algebras is *nondegeneracy* (having no nonzero trivial elements).

5 Special Examples

There were three basic examples of linear Jordan algebras known to Jordan. The first two are the direct parents of Jordan theory: matrix algebras and their hermitian offspring.

Full Example A^+. *It is easy to verify that any associative algebra A can be converted into a Jordan algebra A^+ by forgetting the associative structure but retaining the Jordan products $x^2 := xx, \{x,y\} := xy + yx, U_xy := xyx$.*

Hermitian Example $\mathcal{H}(A,*)$. *In any associative algebra A with involution $*$, the space of hermitian elements $x^* = x$ of A is closed under symmetric products $x^2, xy+yx, xyx$, but not in general under the product xy, and inherits all identities from its full parent, so it forms a Jordan subalgebra of A^+, but not an associative subalgebra of A.*

The full algebra can be swept under the hermitian rug, so the second example swallows the first: if we take A' to be the direct sum of A and its opposite algebra under the exchange involution $(x \oplus y)^* := y \oplus x$, then $\mathcal{H}(A',*)$ is isomorphic to A^+.

The next example is a "sport", an accidental offspring, satisfying the Jordan identity because the square is a linear combination of the element and the unit.

Spin Factor Example $\mathcal{J}(Q,c)$. *Any quadratic form Q on a Φ-module with basepoint c $(Q(c)=1)$ gives rise to a Jordan algebra by $x^2 := T(x)x - Q(x)c$, $\{x,y\} := T(x)y + T(y)x - Q(x,y)c$, $U_x y := Q(x,\bar{y})x - Q(x)\bar{y}$ for $\bar{y} := T(y)c - y$, $T(y) := Q(y,c)$, which lives happily as a Jordan subalgebra inside the associative Clifford algebra of the quadratic form with basepoint.*

These examples provided a rogue's gallery of the algebras the physicists were *not* looking for.

Special Definition. *A Jordan algebra is* **special** *if it can be linearly imbedded in an associative algebra so that the product $x \bullet y$ or $U_x y$ becomes $\frac{1}{2}(xy+yx)$ or xyx, i.e., if it is isomorphic to some Jordan subalgebra of some Jordan algebra A^+, otherwise it is* **exceptional**.

In a special Jordan algebra the algebraic structure is derived from an ambient associative structure via the bullet. What the physicists were looking for, of course, were Jordan algebras where there is no invisible structure xy governing the visible structures $x \bullet y$ and xyx from behind the scenes.

6 Classification

Having settled on the basic axioms for his systems, the third step in Jordan's program was to classify them and find exceptional algebras.

Jordan–von Neumann–Wigner Theorem (1934)[5]. *Every finite-dimensional formally real Jordan algebra is a direct sum of a finite number of simple ideals, and there are five basic types of simple building blocks: four types of hermitian matrix algebras $\mathcal{H}_n(C) := \mathcal{H}(M_n(C),*)$ over the four real composition division algebras $C = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{K}$ (reals, complexes, Hamilton's quaterions, Cayley's octonions) under the conjugate transpose involution (but for $\mathbb{K}$ only $n=3$ is allowed!), together with the spin factors.*

There was only one small surprise in this list, a 27-dimensional structure $\mathcal{H}_3(\mathbb{K})$ which met the Jordan axioms but wasn't hermitian matrices over something associative. A.A. Albert showed that it was indeed an exceptional Jordan algebra (we now call such 27-dimensional exceptional algebras *Albert algebras*). This small creature seemed of no use to physicists, so they abandoned it to the field of algebra, where it surprisingly took root, developing strong symbiotic relationships with the exceptional Lie groups and algebras, exceptional symmetric domains and spaces, and Moufang projective planes. Jordan algebras (including the Albert algebras and spin factors) arise precisely as the coordinates of 3-graded Lie algebras, and hermitian Jordan triple systems arise as the natural models of bounded symmetric domains.

Jacobson extended this classification to linear Jordan algebras which were finite-dimensional over arbitrary fields, then to algebras with d.c.c. on inner ideals [3], and finally to algebras with finite "capacity" [4]. The brilliant young Novosibirsk mathematician Efim Zel'manov then quashed all remaining hopes for such an exceptional system, showing that even in infinite dimensions there are no simple exceptional Jordan algebras other than Albert algebras.

Zel'manov Theorem (1983)[11]. *Every simple linear Jordan algebra of arbitrary dimension is either a hermitian algebra $\mathcal{H}(A,*)$ for a $*$-simple associative algebra A, a spin factor $J(Q,c)$ for a nondegenerate quadratic form Q over a field, or a 27-dimensional Albert algebra $J(N,c)$ determined by a nondegenerate Jordan cubic N over a field. Every prime nondegenerate Jordan algebra is a "form" of one of these three types.*

The same is true of simple quadratic Jordan algebras, except for a few additional wrinkles (ample outer ideals) in characteristic 2 (cf. [6]).

7 The Charge of Illegitimacy

While physicists abandoned the poor orphan child of their theory, the Albert algebra, algebraists adopted it and moved to new territories. A large part of the richness of Jordan theory is due to its ability to handle these exceptional objects, spin objects, and hermitian objects in one algebraic framework.

Actually, the child *should never have been conceived in the first place:* it does *not* obey all the algebraic properties of the Copenhagen model. Jordan was *wrong* in thinking that his axioms had captured the hermitian essence — he overlooked some algebraic properties of hermitian matrices. Firstly, he *missed some algebraic operations* which could not be built from the bullet: the symmetric **n-tad products** $\{x_1,\ldots,x_n\} := x_1\cdots x_n + x_n\cdots x_1$ cannot be expressed in terms of the bullet for $n \geq 4$. In particular, the **tetrads** $\{x_1,x_2,x_3,x_4\} := x_1x_2x_3x_4 + x_4x_3x_2x_1$ were inadvertently excluded from Jordan theory. Their inclusion in the axioms would have excluded *both* the Albert algebras *and* the spin factors, landing back in Copenhagen with nothing but hermitian algebras.

Secondly, Jordan *missed some laws for the bullet* which cannot be derived from the Jordan identity. The smallest of these are Glennie's identities G_8 and G_9 of degree 8 and 9 in 3 variables discovered in 1963, and Thedy's more transparent identity T_{10} of degree 10 in 3 variables discovered in 1987, which are satisfied by all special Jordan algebras, but not by the Albert algebra. Nonzero Jordan polynomials in the free Jordan algebra which vanish on all special algebras are called **special identities** (or **s-identities**). These s-identities are in fact just the algebraic identities satisfied by all hermitian matrices which are *not* consequences of the Jordan axioms. The first thing to say about them is that *there weren't supposed to be any s-identities!* Remember that Jordan's goal was to capture the algebraic behavior of hermitian operators in the Jordan axioms.

A Jordan algebra is **i-special** (identity special) if it satisfies all the identities that special algebras do (all the s-identities), otherwise it is **i-exceptional**. By his own professed principles, Jordan should have excommunicated all i-exceptional algebras.[3]

Not only did the i-exceptional Albert algebra not carry a tetrad operation as hermitian matrices do, but even with respect to its Jordan product it was distinguishable from hermitian matrices by its refusal to obey the s-identities.

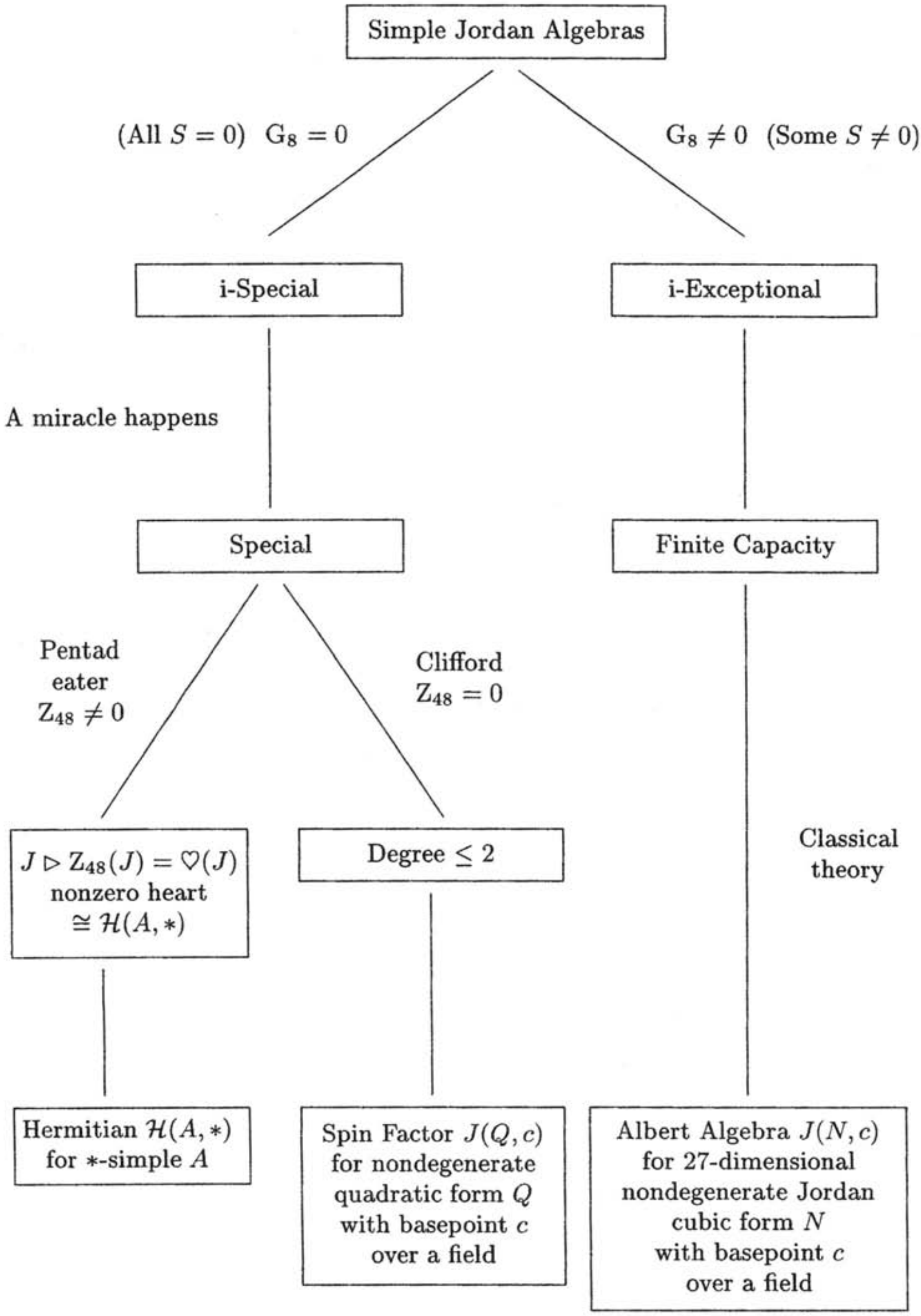
8 The Simple Jordan Evolutionary Tree

We can now show how the s-identity G_8 and Zelmanov's eater identity Z_{48} shape the development of prime and simple Jordan algebras. We present a clean logical flow of evolution, but stress that for technical reasons Zelmanov's detailed *proof* cannot follow such a simple path, and instead wanders along a prime path through primitive rings, big algebraically closed fields, and ultrafilters, before it reaches an Albert, hermitian, or spin home.

The Albert branch

The first branching occurs when a simple Jordan algebra J decides whether to satisfy Glennie's identity G_8 or not. If it does not, it heads inexorably down the slope towards the Albert algebra. Here it is in fact irrelevant *which* s-identity it disobeys: the slightest disobedience (i.e., i-exceptionality) sends it down the Albert evolutionary branch. Indeed, Zel'manov gave a beautiful combinatorial argument to show that *any non-vanishing s-identity f of total degree N* imposes a bound $2N$ on the number of relatively prime inner ideals, which leads to a spectral bound and then to a finite capacity, where Jacobson's classical theory allows only the Albert algebra to be exceptional. The finite degree of the polynomial f turns out to be the *crucial finiteness condition which dooms exceptional algebras to a 27-dimensional life.*

[3]Or perhaps not. Recall that he insisted that all rules imposed should be *physically meaningful,* and no physicist has ever cared in the slightest whether some s-identity was satisfied or not. Even mathematicians (except for those specifically studying the variety of i-special algebras) use s-identities only as tools to show that an algebra is exceptional, then discard them.


Simple Jordan Algebras
(All $S = 0$) $G_8 = 0$
$G_8 \neq 0$ (Some $S \neq 0$)
i-Special
i-Exceptional
A miracle happens
Special
Finite Capacity
Pentad eater $Z_{48} \neq 0$
Clifford $Z_{48} = 0$
$J \triangleright Z_{48}(J) = \heartsuit(J)$ nonzero heart $\cong \mathcal{H}(A, *)$
Degree ≤ 2
Classical theory
Hermitian $\mathcal{H}(A, *)$ for $*$-simple A
Spin Factor $J(Q, c)$ for nondegenerate quadratic form Q with basepoint c over a field
Albert Algebra $J(N, c)$ for 27-dimensional nondegenerate Jordan cubic form N with basepoint c over a field

The special branch

If a simple J does satisfy G_8, it is a miraculous fact that it automatically satisfies *all* s-identities, and hence is i-special. Moreover, another miracle guarantees that a simple i-special algebra doesn't merely *look* special as regards the laws it obeys, it actually *is* special, living inside an associative envelope.[4]

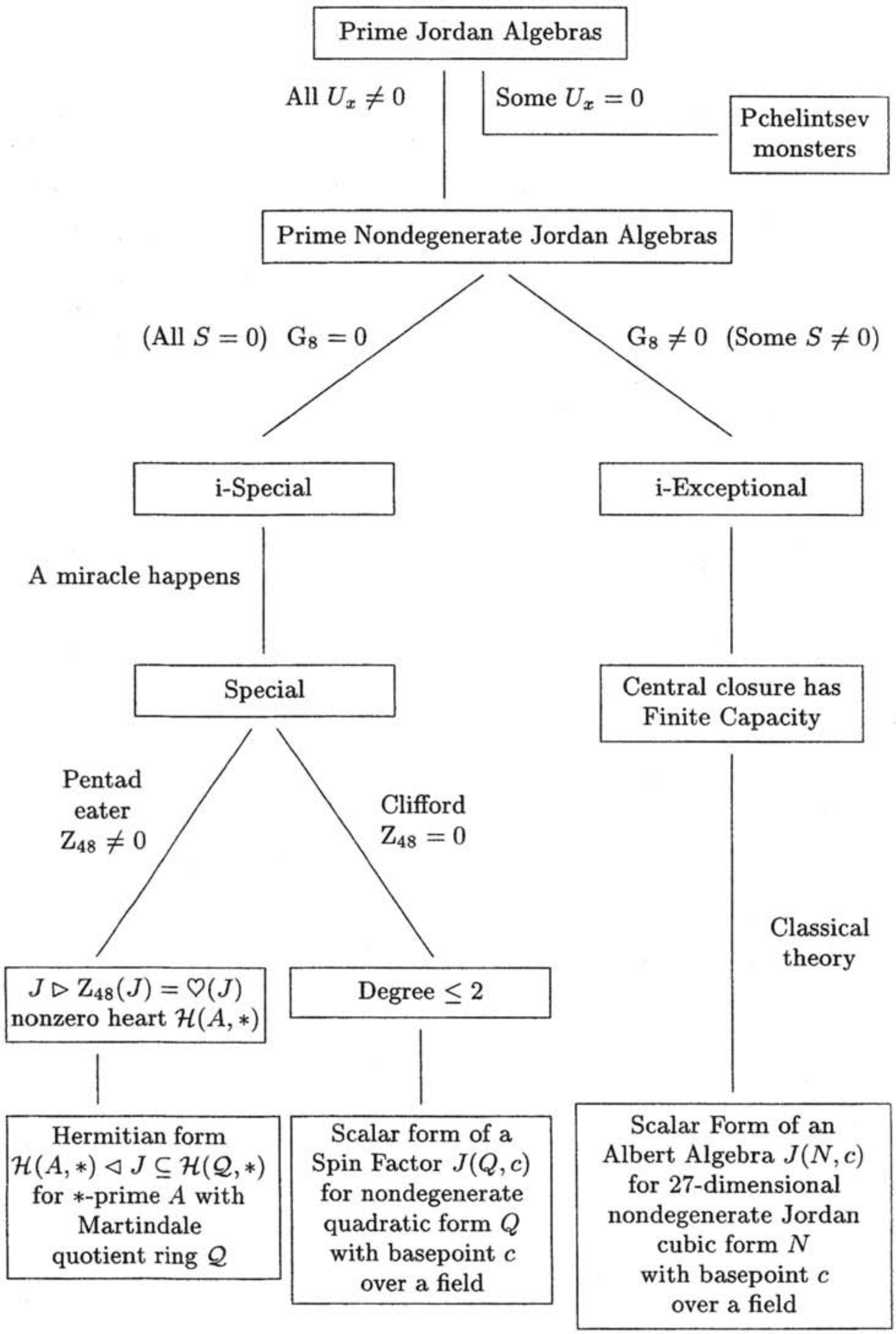
The hermitian branch

Once we have entered the special branch of the Jordan family tree, the second major branching occurs when the algebra decides whether to satisfy Zelmanov's penteater identity Z_{48}. If it does *not*, it is destined to be hermitian. Once more, it is irrelevant which penteater is used: if the penteater ideal $\mathcal{H}_5(J)$ consisting of the values taken on by *all* penteaters is nonzero, it forms the heart $\heartsuit(J)$ (the minimal nonzero ideal, the intersection of all nonzero ideals), which by simplicity must be the entire algebra. The penteater ideal is automatically the algebra of all hermitian elements of an associative algebra with involution, so $J = \heartsuit(J) = \mathcal{H}_5(J) = \mathcal{H}(A, *)$. (In characteristic 2, J may be only an "ample outer ideal" in $\mathcal{H}(A, *)$, for example hermitian matrices over an inseparable field Ω with diagonal entries in $\Phi = \Omega^2$.)

The spin branch

If Z_{48} vanishes identically on J, then miraculously *all* penteaters vanish on J, and $\mathcal{H}_5(J) = 0$. But Z_{48} leads a Jekyll-and-Hyde existence: besides being a penteater, it is a *Clifford identity*, one that does *not* vanish on hermitian matrices of size 3 or greater, guaranteeing that an algebra satisfying Z_{48} has "degree ≤ 2". Once a simple algebra satisfies *some* Clifford identity it satisfies the *standard Clifford identity* $[[x, y]^2, z, w] = 0$ (that squares of commutators lie in the center), and lives inside the Clifford algebra of a quadratic form over a field. Thus $J = J(Q, c)$ is a spin factor. (For small quadratic forms in characteristic 2, J may be only an "ample outer ideal" in $J(Q, c)$.)

[4] It would be desirable to have a direct proof of this, rather than the current argument which makes an end run through the structure theory, concluding afterwards that the miracle must have occurred. One would like to know that the specializer ideal is degenerate modulo the i-specializer ideal.


Prime Jordan Algebras
All $U_x \neq 0$
Some $U_x = 0$
Pchelintsev monsters
Prime Nondegenerate Jordan Algebras
(All $S = 0$) $\mathrm{G}_8 = 0$
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Scalar form of a Spin Factor $J(Q, c)$ for nondegenerate quadratic form Q with basepoint c over a field
Scalar Form of an Albert Algebra $J(N, c)$ for 27-dimensional nondegenerate Jordan cubic form N with basepoint c over a field

9 The Prime Jordan Evolutionary Tree

Zelmanov's proof in fact derived the structure of simple algebras from that of the more general *prime* algebras (those having no orthogonal ideals). In the simple case, a highly nontrivial proof shows that simple Jordan algebras are *automatically* nondegenerate. This is not true in the prime case: there do exist prime degenerate algebras, bearing no genetic relation to the standard three types of Jordan algebra (hermitian, spin, and Albert). These are named **Pchelintsev monsters**, in honor of their discoverer. We avoid this evolutionary dead end, and return to the Garden of Eden inhabited by prime *nondegenerate* Jordan algebras.

Exactly as in the simple case, the first prime branching occurs when a prime nondegenerate Jordan algebra J decides whether to satisfy Glennie's identity G_8 or not. If it does not, it is i-exceptional, and again a non-vanishing s-identity f leads to finite capacity for the central closure and life imprisonment as an Albert algebra. If a prime J does satisfy G_8, it is again miraculously i-special and then special.

Once inside the special branch, the second major branching occurs when the algebra decides whether to satisfy Zelmanov's eater identity Z_{48}. Once more, it is irrelevant which penteater is used: if the penteater ideal $\mathcal{H}_5(J)$ consisting of the values taken on by *all* hearty pentad-eaters is nonzero, it forms the heart $\heartsuit(J)$, and J is trapped between its heart and the "heart quotient": $\mathcal{H}(A,*) = \mathcal{H}_5(J) = \heartsuit(J) \lhd J \subseteq \mathcal{H}(\mathcal{Q}(A),*)$ where $\mathcal{Q} = \mathcal{Q}(A)$ is the Martindale quotient ring of A. Thus J is a *hermitian form* of $\mathcal{H} = \mathcal{H}(\mathcal{Q},*)$, not in the sense of becoming $\mathcal{H}$ under scalar extension, $J_\Omega = \mathcal{H}$, but rather in the sense of having $\mathcal{H}$ as "ring of quotients" as in the Goldie theory of prime noetherian associative rings.

If Z_{48} vanishes identically on J, then again *all* pentad-eating polynomials amazingly vanish on J, and the Clifford identity Z_{48} guarantees that J has "degree ≤ 2" and is a *scalar form* of a spin factor, $J_\Omega = J(Q,c)$.

Just as the structure of simple algebras is actually derived from that of prime algebras, so the structure of prime algebras is actually derived from that of primitive algebras over algebraically closed fields. Here an algebra is *primitive* if it has a proper "modular" inner ideal P which is "dense" in that it complements all nonzero ideals $I \lhd J$, $I + P = J$. In the presence of nondegeneracy, the prime algebra J may be imbedded in the direct product $\prod_X J_x$ of primitive algebras J_x over big algebraically closed fields, where each J_x comes in one of three flavors: Albert, hermitian, or spin. The supports of nonzero elements of the prime subalgebra J induce a filter on the index set X, which can be refined to an ultrafilter $\mathcal{F}$; a prime trichotomy theorem guarantees the ultraproduct $\left(\prod_X J_x\right)/\mathcal{F}$ is a product of factors *all of the same flavor*, and the prime subalgebra J is a form of an Albert, hermitian, or spin algebra.

10 Glennie's Identities

We now turn to examine in more detail these identities which shape the course of Jordan evolution.

Glennie's Identities[2]. G_8, G_9 *are the Jordan polynomials*

$$\mathrm{G}_n := \mathrm{H}_n(x,y,z) - \mathrm{H}_n(y,x,z)$$

of degrees $n = 8, 9$ *expressing the symmetry in* x, y *of the products* H_n *(which reduce in associative algebras to the symmetric octad* $\mathrm{H}_8 = \{x,y,z,y,x,z,x,y\}$ *and nonad* $\mathrm{H}_9 = \{x,z,x^2,z,y^2,z,y\}$*) :*

$$\begin{aligned}\mathrm{H}_8(x,y,z) &:= \{U_xU_yz,\ z,\ \{x,y\}\} - U_xU_yU_z(\{x,y\}),\\ \mathrm{H}_9(x,y,z) &:= \{U_xz,\ U_{y,x}U_zy^2\} - U_xU_zU_{x,y}U_y(z).\end{aligned}$$

These two are closely intertwined: G_8 is a partial linearization of G_9,

$$\mathrm{G}_8(x,y,z) = \partial_1\mathrm{G}_9(x,y,z)|_z,$$

while

$$2\mathrm{G}_9(x,y,z) = V_z\mathrm{G}_8(x,y,z) + V_x\mathrm{G}_8(z,y,x) - V_y\mathrm{G}_8(z,x,y),$$

so both generate essentially the same endvariant ideal.

Open Question. *Can anyone remove the* 2 *here to get* G_9 *itself built out of* G_8*??*

The quick modern proof that the Albert algebra is i-exceptional is to show that $\mathrm{G}_8(x,y,z)$ or $\mathrm{G}_9(x,y,z)$ does not vanish on $\mathcal{H}_3(K)$ for a judicious choice of x, y, z.

11 Commutators

Zel'manov has shown us that commutators $[x,y]$ are crucial ingredients of Jordan theory. While they don't exist in a Jordan algebra itself, they lead an ethereal existence lurking at the fringe of the Jordan algebra, just waiting to manifest themselves. They do leave footprints: many commutator products do exist within the Jordan algebra. In associative algebras we have

$$\begin{aligned}[[x,y],z] &= (xyz+zyx)-(yxz+zxy)\\ [x,y]^2 &= (xy-yx)^2 = (xy+yx)^2 - 2(xyyx+yxxy)\\ [x,y]z[x,y] &= (xy+yx)z(xy+yx) - 2(xyzyx+yxzxy),\\ [[x,y]^3,z] &= \Big[[x,y],\Big[[x,y],[[x,y],z]\Big]\Big] + 3\Big[[x,y],[x,y]z[x,y]\Big],\end{aligned}$$

which motivates the following Jordan definitions:

$$\begin{aligned}
[[x,y],z] &:= (V_{x,y} - V_{y,x})(z) = [V_x, V_y](z) =: D_{x,y}(z),\\
[x,y]^2 &:= \{x,y\}^2 - 2\big(U_x(y^2) + U_y(x^2)\big),\\
U_{[x,y]} &:= U_{\{x,y\}} - 2\{U_x, U_y\},\\
[[x,y]^3, z] &:= \big(D^3_{x,y} + 3D_{x,y}U_{[x,y]}\big)(z) := D'_{x,y}(z).
\end{aligned}$$

Note that the usual inner derivation $D_{x,y}$ is a manifestation of $Ad_{[x,y]}$, and $D'_{x,y} = D^3_{x,y} + 3D_{x,y}U_{[x,y]}$ is a manifestation of $Ad_{[x,y]^3}$, where the ghostly $[x,y]$ and $[x,y]^3$ themselves become visible only in associative envelopes of special Jordan algebras. We may think of commutators as some Copenhagenish emanations that surround all Jordan algebras.

Tip of the Day: Invest in Commutators !

12 Thedy's Identity

A more user-friendly identity was discovered by Thedy during his study of right alternative algebras.

Thedy's Identity[10]. T_{10} *is the Jordan operator polynomial of degree* 10

$$\mathrm{T}_{10}(x,y,z) := U_{U_{[x,y]}(z)} - U_{[x,y]}U_zU_{[x,y]}.$$

This is just a "fundamental formula" or "structural condition" for the U-operator of the "commutator" $[x,y]$; acting on an element w, this produces an element polynomial of degree 11

$$\mathrm{T}_{11}(x,y,z,w) := \mathrm{T}_{10}(x,y,z)(w) = U_{U_{[x,y]}(z)}(w) - U_{[x,y]}U_zU_{[x,y]}(w).$$

Thedy's Identity Unmasked [9]. *Thedy's polynomial vanishes identically in characteristic* 2, *since it has the form*

$$\mathrm{T}_{10}(x,y,z) = 2\mathrm{S}_{10}(x,y,z) - 2\mathrm{S}_{10}(y,x,z)$$

measuring symmetry in x and y of a more basic, but uglier, s-identity

$$\mathrm{S}_{10} := \{U_x, U_y, U_z, W_{x,y}\} - U_{U_xU_y(z),W_{x,y}(z)}$$

in terms of the operator

$$W_{x,y} := U_{\{x,y\}} - \{U_x, U_y\} = U^2_{x,y} - U_{x^2,y^2}$$

(which reduces to the pentad $\{x,y,\cdot,x,y\}$ in special algebras).

13 Shestakov's Identities

In 1999 Ivan Shestakov discovered that Glennie's identities could be rewritten in a very memorable form using commutators, namely that $Ad_{[x,y]^3}$ is a derivation. The derivation rule for squares is G_8, from the square we get the bullet product, and hence that $Ad_{[x,y]^3}$ is a derivation on *any* auxiliary product, for example the cube and the U-product.[5]

Shestakov's Identities. *III_8, III_9 are the Jordan polynomials*

$$\mathit{III}_8(x,y,z) := \; [[x,y]^3,\, z^2] - \{z,\, [[x,y]^3,\, z]\},$$
$$\mathit{III}_9(x,y,z) := \; [[x,y]^3,\, z^3] - \{z^2, [[x,y]^3, z]\} - U_z\big([[x,y]^3, z]\big).$$

Shestakov's Identities Unmasked. *Shestakov's polynomial III_8 vanishes identically in characteristics 2 and 3, since it has the form*

$$\mathit{III}_8(x,y,z) = 6\mathrm{G}_8(x,y,z).$$

Shestakov's polynomial III_9 vanishes identically in characteristic 3, since it has the form

$$\mathit{III}_9(x,y,z) = 3\mathrm{S}_9(x,y,z).$$

for a more basic, but uglier, s-identity

$$\mathrm{S}_9(x,y,z) := \{D(z), D(z), D(z)\} + \{z, D(z), D^2(z)\} + \{z, D^2(z), D(z)\}$$
$$+\{D(z), z, D^2(z)\} + DC(z^3) - (V_{z^2} + U_z)DC(z)$$

(abbreviating $D_{x,y}, U_{[x,y]}$ by D, C). We have

$$6\mathrm{S}_9(x,y,z) = 6\Big(\mathrm{G}_8(x,y;z,z^2) + \{z, \mathrm{G}_8(x,y,z)\}\Big).$$

Here $G_8(x,y;z,w)$ denotes the polarization of G_8 in the quadratic variable z, the coefficient $\partial^z_w G_8(x,y,z)$ of λ in the expansion of $G_8(x,y,z+\lambda w)$.

Open Question. *Can we cancel 6 to get S_9 built out of G_8?*

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[5]Though written by an author I otherwise admire, the book [7] is incorrect when it states on p.350: "The derivation rule for cubes ... is just G_9. This makes it crystal clear that G_8 implies G_9." The derivation rule III_9 for cubes is indeed an s-identity of degree 9, but it is *not* $G_9(x,y,z)$; there seem to be 6 different s-identities of degree 9, and it is not yet clear who they are and which one III_9 is.

14 Pentad-eating Clifford Polynomials

The polynomial enzyme which breaks down special Jordan algebras into hermitian and spin types is Zelmanov's penteater. A Jordan polynomial $f(X) = f(x_1, \ldots, x_n)$ is an **n-tad eater** if it converts n-tads into 3-tads (which are Jordan products): $\{t_1, \ldots, t_{n-1}, f(X)\} = \sum\{g_1, g_2, g_3\}$ for Jordan polynomials $g_i = g_i(t_1, \ldots, t_{n-1}, X)(i = 1, 2, 3)$. A **hearty** n-tad eater is one that eats anything resembling an n-tad. A **penteater** is a hearty pentad (5-tad) eating polynomial.

A **Clifford polynomial** is one which does not vanish on 3×3 matrices $\mathcal{H}_3(\Phi)$ (so it vanishes only on algebras of degree 2 or less). The standard Clifford polynomial is $[[[x,y]^2, z], w]$, whose vanishing means that squares of commutators lie in the center.

Zel'manov Penteater. Z_{48} *is the Jordan polynomial of degree* 48 *in* 12 *variables*

$$\mathrm{Z}_{48}(X,Y,Z,W) := [[p_{16}(x_1,y_1,z_1,w_1), p_{16}(x_2,y_2,z_2,w_2)], p_{16}(x_3,y_3,z_3,w_3)]$$

$$\left(p_{16}(x,y,z,w) := [[D^2_{x,y}(z)^2, D_{x,y}(w)], D_{x,y}(w)] \right)$$

This is a hearty pentad-eating Clifford polynomial; it is a slight improvment of his original polynomial $\mathrm{Z}_{2160} = p_{12}^{180}$. I don't know what the smallest penteater is.

15 Peirce s-Identities

While we are on the subject of identities, let me inject a remark about Peirce identities. These are a special case of the *generalized polynomial identities*, such as those studied in associative algebras by Jerry Martindale.

Peirce Identity Definition. *A* Peirce s-identity *is a Jordan polynomial* $f(e_1, \ldots, e_n, x_{i_1 j_1}, \ldots, x_{i_r, j_r})$ *in the free Jordan algebra with* n *idempotents which vanishes on all special Jordan algebras, but not all Jordan algebras, having Peirce decomposition determined by orthogonal family* $e_1, \ldots, e_n$ *of idempotents with elements* x_{ij} *from the Peirce spaces* J_{ij}.

Example. *For* $n = 3$,

$$\mathrm{PRS}_5(x_{12}, y_{12}, x_{13}, y_{13}, z_{23}) = [V_{x_{12},y_{12}}, V_{x_{13},y_{13}}](z_{23})$$

is a Peirce s-identity which does not vanish on the Albert algebra:

$$\mathrm{PRS}_5(a[21], 1[12], c[13], 1[31], b[23]) = \big(a(bc) - (ab)c\big)[23]$$

where there are non-associating octonions a, b, c.

This is by far the most painless way to show that the Albert algebra is exceptional, even that it is Peirce-i-exceptional (though this does not quite show it is i-exceptional). Very little is known about Peirce s-identities, or about the free Jordan algebra with n orthogonal idempotents.

Let me close with one other remark about the basic concept of speciality (living in an associative context or having "associative genes"): in general, speciality depends on our choice of scalars, on our categorical perspective (the category of Φ-algebras or Ω-algebras). If an Ω-algebra J is Φ-special for $\Phi \subset \Omega$ a subring of scalars, it need not also be Ω-special [8]. One suspects this happens only for unreasonable algebras (all known examples involve degeneracy and 2-torsion).

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