São Paulo Journal of Mathematical Sciences **4**, 1 (2010), 109–120

Gorenstein Quivers

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Abstract. We introduce a notion of *Gorenstein quiver* associated with a Gorenstein matrix. We study properties of such quivers. In particular, we show that any such quiver is strongly connected and simply laced. We use Perron-Frobenius theory of non-negative matrices for characterization of isomorphic Gorenstein quivers.

1. Introduction

The notion of exponent and Gorenstein matrix has origin in ring theory. It is important in the study of Gorenstein rings considered by H.Bass in 1963 (see [1]). In particular, these concepts are relevant to the study of Gorenstein tiled orders [5, Ch.7].

In this paper we introduce a notion of *Gorenstein quiver* associated with a Gorenstein matrix and study the properties of these quivers. In Section 2 we give a short survey of main results on semiprime right Noetherian semiperfect and semidistributive rings which lead to a concept of a Gorenstein matrix and preliminary results on Gorenstein matrices. In Section 3 we recall classical theorems of Perron and Frobenius on non-negative matrices which play important role in our characterization of isomorphic Gorenstein quivers. Finally, Section 4 contains our main results. Here we define a Gorenstein quiver associated with every Gorenstein matrix. We show that any such quiver is strongly connected and simply laced. We also give a characterization of isomorphic Gorenstein quivers. At the end we give some examples of Gorenstein quivers.

2. Preliminaries

2.1. **SPSD-rings.** For convenience of the reader we recall main result on semiprime right Noetherian semiperfect and semidistributive rings (see [4, Ch.14]). We write SPSD-ring for semiperfect and semidistributive ring (see [4, Ch.14]).

Definition 2.1. *A ring is called* semimaximal *if it is a semiperfect semiprime right Noetherian ring such that for each local idempotent $e \in A$ the ring eAe is a discrete valuation ring (not necessary commutative).*

The following is a decomposition theorem for semiprime right Noetherian SPSD-rings.

Theorem 2.1. *The following conditions for a semiperfect right Noetherian SPSD-ring are equivalent:*

(a) the ring A is semidistributive;

(b) the ring A is a direct product of a semisimple Artinian ring and a semimaximal ring;

Theorem 2.2. *Each semimaximal ring is isomorphic to a finite direct product of prime rings of the following form:*

$$\begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \cdots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \cdots & \pi^{\alpha_{2n}}\mathcal{O} \\ \cdots & \cdots & \cdots & \cdots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}$$

where $n \geqslant 1$, $\mathcal{O}$ is a discrete valuation ring with a prime element π, and α_{ij} are integers such that $\alpha_{ij} + \alpha_{jk} \geqslant \alpha_{ik}$ for all i, j, k ($\alpha_{ii} = 0$ for any i).

Definition 2.2. *A matrix $\mathcal{E} = (\alpha_{ij})$ is called* exponent matrix *if $\mathcal{E}$ satisfies the following two conditions:*

- $\alpha_{ii} = 0$ *for* $i = 1, \ldots, n$;
- $\alpha_{ij} + \alpha_{jk} = \alpha_{ik}$ *for* $i, j, k = 1, \ldots, n$.

An exponent matrix $\mathcal{E}$ is called reduced exponent matrix *if $\alpha_{ij} + \alpha_{ji} > 0$ for $i \neq j$.*

Denote by $M_n(B)$ a ring of all $(n \times n)$-matrices with elements from a ring B. Let $\mathcal{O}$ be a discrete valuation ring with prime element π and $\mathcal{M} = \pi\mathcal{O} = \mathcal{O}\pi$ is the unique maximal ideal of $\mathcal{O}$, D is the classical division ring of fractions of $\mathcal{O}$.

Denote by $A = \{\mathcal{O},\, \mathcal{E} = (\alpha_{ij})\}$ the following subring of $M_n(D)$:

$$A = \begin{pmatrix} \mathcal{O} & \pi^{\alpha_{12}}\mathcal{O} & \dots & \pi^{\alpha_{1n}}\mathcal{O} \\ \pi^{\alpha_{21}}\mathcal{O} & \mathcal{O} & \dots & \pi^{\alpha_{2n}}\mathcal{O} \\ \dots & \dots & \dots & \dots \\ \pi^{\alpha_{n1}}\mathcal{O} & \pi^{\alpha_{n2}}\mathcal{O} & \dots & \mathcal{O} \end{pmatrix}.$$

A ring A is a semiperfect and semidistributive prime Noetherian ring with nonzero Jacobson radical (tiled order), see [4, Ch.14].

Let A be a semiperfect ring with the Jacobson radical R. A ring A is called *reduced* if A/R is a direct product of division rings. In particular, a tiled order $A = \{\mathcal{O}, \mathcal{E} = (\alpha_{ij})\}$ is reduced if and only if its exponent matrix $\mathcal{E} = (\alpha_{ij})$ is reduced.

2.2. **Gorenstein tiled orders and Gorenstein matrices.** In this section we collect necessary statements about Gorenstein matrices.

Let $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z})$, where $\mathbb{Z}$ is the ring of integers.

Definition 2.3. *A reduced exponent matrix $\mathcal{E}$ is called* Gorenstein matrix *if there exists a permutation τ of the set $\{1, \dots, n\}$ such that $\alpha_{ij} + \alpha_{j\tau(i)} = \alpha_{i\tau(i)}$ for all i and j.*

Theorem 2.3. [5, Ch.7] *The following properties for reduced tiled order $A = \{\mathcal{O}, \mathcal{E} = (\alpha_{ij})\}$ are equivalent:*

(a) inj.dim $A_A = 1$;

(b) inj.dim ${}_AA = 1$;

(c) the exponent matrix $\mathcal{E} = (\alpha_{ij})$ is Gorenstein.

Recall that a commutative ring is called Gorenstein if its injective dimension is finite.

If a reduced tiled order $A = \{\mathcal{O}, \mathcal{E} = (\alpha_{ij})\}$ satisfies the conditions of Theorem 2.3 then it will be called *Gorenstein tiled order.* In particular, from [5, Ch.7] we obtain the following statement.

Corollary 2.1. *If A is a reduced Gorenstein tiled order then all rings $B_k = A/\pi^k A$ are Frobenius for $k \geqslant 1$. If $\mathcal{O}/\pi\mathcal{O}$ is a finite ring, then all B_k are finite Frobenius rings.*

Theorem 2.3 and Corollary 2.1 indicate the importance of the study of Gorenstein matrices.

2.3. **Quivers.** Following P.Gabriel a *quiver* is a finite directed graph.

A quiver $Q = (VQ, AQ, s, e)$ is a finite directed graph which consists of finite sets VQ and AQ and two mappings $s, e : AQ \to VQ$. The elements of VQ are called **vertices** (or **points**), and those of AQ are called **arrows**.

Usually, the set of vertices VQ will be a set $\{1, 2, \ldots, n\}$. We say that each arrow $\sigma \in AQ$ starts at the vertex $s(\sigma)$ and ends at the vertex $e(\sigma)$. The vertex $s(\sigma)$ is called the **start** (or **initial**, or **source**) **vertex** and the vertex $e(\sigma)$ is called the **end** (or **target**) vertex of σ.

A quiver without multiple arrows and multiple loops is called a *simply laced* quiver.

Assume that we have t_{ij} arrows from the vertex i to the vertex j. The $(n \times n)$-matrix $[Q] = (t_{ij})$ is called the *adjacency* matrix of the quiver Q. A quiver Q is simply laced if and only if its adjacency matrix $[Q]$ is $(0, 1)$-matrix.

Let e_{ij}, $i, j = 1, \ldots, n$, be the matrix units in $M_n(\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers, $B = \sum\limits_{i,j=1}^{n} b_{ij} e_{ij} \in M_n(\mathbb{R})$.

Recall that the quiver $Q = Q(B)$ of a matrix $B = (b_{ij})$ is the simply laced quiver with $VQ = \{1, \ldots, n\}$ and there exists the arrow $\sigma : i \to j$ if and only if $b_{ij} \neq 0$.

Let $\tau : i \to \tau(i)$ is a permutation of $\{1, \ldots, n\}$. A matrix $P_\tau = \sum\limits_{i=1}^{n} e_{i\tau(i)}$ is called a *permutation matrix* corresponding to τ.

Two quivers Q_1 and Q_2 are called *isomorphic* ($Q_1 \simeq Q_2$) if there is a bijective correspondence between vertices and arrows such that starts and ends of corresponding arrows map into each other. In this case there exists a permutation matrix P_τ such that $[Q_2] = P_\tau^T [Q_1] P_\tau$, where T denotes the transpose.

Conversely, if $[Q_2] = P_\tau^T [Q_1] P_\tau$, then $Q_1 \simeq Q_2$.

Let $[Q]$ be the adjacency matrix of a quiver Q and $[Q] \in M_n(\mathbb{C})$, where $\mathbb{C}$ is the field of the complex numbers.

Let $\vec{z}^T = (z_1, \ldots, z_n)^T \in \mathbb{C}^n$ be a right eigenvector of $[Q]$ with an eigenvalue λ, i.e.,

$$[Q]\vec{z}^T = \lambda \vec{z}^T$$

and $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{C}^n$ be a left eigenvector of $[Q]$ with an eigenvalue μ, i.e.,

$$\vec{u}\,[Q] = \lambda \vec{u}.$$

Let τ be a permutation of the set $\{1, \ldots, n\}$ and $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{C}^n$.

Denote by $\vec{a}_\tau$ the n-dimensional vector which is obtained from $\vec{a} = (a_1, \ldots, a_n)$ by the permutation of its coordinates by the rule $\vec{a}_\tau = (a_{\tau(1)}, \ldots,$

$a_{\tau(n)})$. Two n-dimensional vectors $\vec{a}$ and $\vec{b}$ are called *equivalent* if $\vec{b} = \vec{a}_\tau$ for some permutation τ.

Proposition 2.1. *Let Q_1 and Q_2 are two isomorphic quivers. Let $\vec{a}^T$ be the right eigenvector of the matrix $[Q_2]$ with the eigenvalue λ. Then the vector $\vec{a}_\tau{}^T$ is the right eigenvector of the matrix $[Q_1]$ with the same eigenvalue λ. If $\vec{b}^T$ is right eigenvector of $[Q_1]$ with eigenvalue λ then $\vec{b}^T_{\tau^{-1}}$ is the right eigenvector of $[Q_2]$ with eigenvalue λ.*

Proof. The equality $P_\tau[Q_2] = [Q_1]P_\tau$ holds. Let $\vec{a}^T$ be an eigenvector of the matrix $[Q_2]$ with eigenvalue λ. Then $P_\tau[Q_2]\vec{a}^T = \lambda P_\tau\vec{a}^T = \lambda\vec{a}_\tau{}^T = [Q_1]P_\tau\vec{a}^T = [Q_1]\vec{a}_\tau{}^T$, i.e. $\vec{a}_\tau{}^T$ is eigenvector of the matrix $[Q_1]$.

Let $\vec{b}$ be an eigenvector of the matrix $[Q_1]$ with eigenvalue λ. From $[Q_2]P_\tau^{-1} = P_{\tau_{-1}}[Q_1]$ follows $[Q_2]P_{\tau^{-1}}\vec{b}^T = P_{\tau^{-1}}[Q_1]\vec{b}^T = \lambda P_{\tau^{-1}}\vec{b}^T = \lambda\vec{b}^T_{\tau^{-1}} = [Q_2]b^T_{\tau^{-1}}$. □

Definition 2.4. *The* characteristic polynomial $\chi_Q(x)$ *of the quiver Q, is called the characteristic polynomial of the matrix $[Q]$, i.e., $\chi_Q(x) = det(xE - [Q])$.*

Obviously, if $Q_1 \simeq Q_2$, then $\chi_{Q_1}(x) = \chi_{Q_2}(x)$.

Recall that *path* from the vertex i to the vertex j of the quiver Q is called the a sequence of arrows $\sigma_1 \ldots \sigma_r$ such that the start vertex of each arrow σ_m coincides with the end vertex of the previous one σ_{m-1} for all m, $1 < m \leqslant r$ and moreover, the vertex i is the start vertex of σ_1, while the vertex j if the end vertex j is the end vertex of σ_r. The number r of arrows is called *the length of the path.*

Definition 2.5. *Let Q be a quiver and $VQ = \{1, \ldots, n\}$. If $n \geqslant 2$, Q is called* strongly connected *if for any two vertices there exists a path from one to another.*

By convention a one-point quiver will be considered a strongly connected quiver.

3. Perron and Frobenius theorems.

In this section we recall classical theorems of Perron and Frobenius. Recall that a matrix $B \in M_n(\mathbb{R})$ is called *permutationally reducible* if there exists a permutation matrix P_τ such that

$$P_\tau^T B P_\tau = \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix},$$

where B_1 and B_2 are square matrices of order less that n. Otherwise, the matrix B is called *permutationally irreducible.*

From the equality

$$D_n \begin{pmatrix} B_1 & B_{12} \\ 0 & B_2 \end{pmatrix} D_n = \begin{pmatrix} B_1^{(1)} & 0 \\ B_{21} & B_2^{(2)} \end{pmatrix}$$

it follows that B is permutationally reducible if and only if there exists a permutation matrix P_ν such that

$$P_\nu^T B P_\nu = \begin{pmatrix} B_1^{(1)} & 0 \\ B_{21} & B_2^{(2)} \end{pmatrix}, \text{ where } D_n = \sum_{i=1}^{n} e_{i,n-i+1}$$

and $B_1^{(1)}$ and $B_2^{(2)}$ are square matrices of order less that n.

Proposition 3.1. [4, §11.3] *A matrix B is permutationally irreducible if and only if the simply laced quiver $Q(B)$ is strongly connected.*

A vector $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ is called **positive** if $y_i > 0$ for $i = 1, \ldots, n$. The number $\|\vec{y}\| = \sqrt{y_1^2 + \ldots + y_n^2}$ is called the *norm* of vector $\vec{y}$.

We have the following well-known result.

Theorem 3.1 (Perron theorem)**.** [6] *A positive matrix $A = (a_{ij})$ $(i, j = 1, \ldots, n)$ always has a real and positive eigenvalue r which is a simple root of the characteristic equation and which is larger that the absolute values of all other eigenvalues. To this maximal eigenvalue r there corresponds a positive eigenvector $z = (z_1, z_2, \ldots, z_n)$ of A.*

A positive matrix is a special case of a permutationally irreducible non-negative matrix. Frobenius generalized the Perron theorem by investigating the spectral properties of permutationally irreducible non-negative matrices.

Theorem 3.2 (Frobenius theorem)**.** [2] *A permutationally irreducible non-negative matrix $A = (a_{ij})$ $i, j = 1, \ldots, n$ always has a positive eigenvalue r which is a simple root of the characteristic equation. The absolute values of all the other eigenvalues do not exceed r. To the maximal eigenvalue r there corresponds a positive eigenvector.*

Moreover, if A has h eigenvalues $\lambda_0 = r, \lambda_1, \ldots, \lambda_{h-1}$ of absolute value r, then these numbers are all distinct and are roots of the equation

$$\lambda^h - r^h = 0.$$

More generally: The whole spectrum $\lambda_0, \lambda_1, \dots, \lambda_{h-1}$ of A, regarded as a system of points in the complex λ-plane, goes over into itself under a rotation of the plane by the angle $2\pi/h$. If $h > 1$, then, by means of a permutation, A can be brought into the following block cyclic form:

$$A = \begin{pmatrix} 0 & A_{12} & 0 & \dots & \dots & 0 \\ 0 & \ddots & A_{23} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \ddots & A_{h-1,h} \\ A_{h1} & 0 & \dots & \dots & 0 & 0 \end{pmatrix},$$

where there are square blocks along the main diagonal.

A strongly connected quiver Q is called primitive if its adjacency matrix $[Q]$ has only one eigenvalue with maximal absolute value r, otherwise Q is called imprimitive.

4. Gorenstein quivers

In this main section we discuss the properties of Gorenstein quivers.

We will need the following definition.

Definition 4.1. *Let Q be a strongly connected quiver with the adjacency matrix $[Q]$. The maximal positive eigenvalue r is called the index of Q ($r = \operatorname{inx} Q$).*

Remark 4.1. *Let*

$$s_i = \sum_{j=1}^{n} a_{ij} \quad (i = 1, 2, \dots, n), \;\; s = \min_{1 \leqslant i \leqslant n} s_i, \;\; S = \max_{1 \leqslant i \leqslant n} s_i.$$

Then for a permutationally irreducible matrix $A \geqslant 0$

$$s \leqslant r \leqslant S,$$

and the equality sign on the left or the right of r holds for $s = S$ only; i.e., they hold only when all the row-sums $s_1, s_2, \dots, s_n$ are all equal.

Remark 4.2. *A permutationally irreducible matrix $A \geqslant 0$ cannot have two linearly independent positive eigenvectors with the maximal real eigenvalue r.*

Let $\mathcal{E} = (\alpha_{ij})$ be a reduced exponent matrix. Set $\mathcal{E}^{(1)} = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $i \neq j$ and $\beta_{ii} = 1$ for $i = 1\dots,n$. Also set $\mathcal{E}^{(2)} = (\gamma_{ij})$, where $\gamma_{ij} = \min\limits_{1\leqslant k\leqslant n}(\beta_{ik} + \beta_{ki})$. Obviously, $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is a $(0,1)$-matrix.

Theorem 4.1. [4, Ch.14] *The matrix $[Q] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}$ is the adjacency matrix of the strongly connected simply laced quiver $Q = Q(\mathcal{E})$.*

Definition 4.2. *A quiver Q is called Gorenstein if it is the quiver of a Gorenstein matrix.*

We immediately obtain from Theorem 4.1 the following property of Gorenstein quivers.

Corollary 4.1. *A Gorenstein quiver Q is strongly connected simply laced quiver.*

Eigenvector of a matrix $[Q]$ is also called an eigenvector of the quiver Q.

Corollary 4.2. *A Gorenstein quiver Q has a positive eigenvector.*

Example 4.1. *Consider the matrix*

$$\mathcal{E}_{12} = \begin{pmatrix} 0 & 6 & 4 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 3 & 3 \\ 6 & 0 & 4 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 3 & 3 \\ 2 & 2 & 0 & 6 & 4 & 4 & 2 & 2 & 4 & 4 & 2 & 2 \\ 2 & 2 & 6 & 0 & 4 & 4 & 2 & 2 & 4 & 4 & 2 & 2 \\ 2 & 2 & 2 & 2 & 0 & 6 & 4 & 4 & 4 & 4 & 2 & 2 \\ 2 & 2 & 2 & 2 & 6 & 0 & 4 & 4 & 4 & 4 & 2 & 2 \\ 3 & 3 & 4 & 4 & 2 & 2 & 0 & 6 & 4 & 4 & 4 & 4 \\ 3 & 3 & 4 & 4 & 2 & 2 & 6 & 0 & 4 & 4 & 4 & 4 \\ 4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 6 & 4 & 4 \\ 4 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 6 & 0 & 4 & 4 \\ 3 & 3 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 0 & 6 \\ 3 & 3 & 4 & 4 & 4 & 4 & 2 & 2 & 2 & 2 & 6 & 0 \end{pmatrix}$$

This is the Gorenstein matrix with the following permutation

$$\sigma(\mathcal{E}_{12}) = (1,\, 2)(3,\, 4)(5,\, 6)(7,\, 8)(9,\, 10)(11,\, 12).$$

Now we construct the Gorenstein quiver $Q(\mathcal{E})$ of $\mathcal{E}_{12}$.

$\mathcal{E}_{12}^{(1)} = E_{12} + \mathcal{E}_{12}$, *where E_{12} is the identity 12×12-matrix,*

$$\mathcal{E}_{12}^{(2)} = \begin{pmatrix} 2 & 6 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 4 & 4 \\ 6 & 2 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 4 & 4 \\ 3 & 3 & 2 & 6 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 \\ 3 & 3 & 6 & 2 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 6 & 4 & 4 & 4 & 4 & 3 & 3 \\ 3 & 3 & 3 & 3 & 6 & 2 & 4 & 4 & 4 & 4 & 3 & 3 \\ 4 & 4 & 4 & 4 & 3 & 3 & 2 & 6 & 5 & 5 & 4 & 4 \\ 4 & 4 & 4 & 4 & 3 & 3 & 6 & 2 & 5 & 5 & 4 & 4 \\ 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 2 & 6 & 4 & 4 \\ 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 6 & 2 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 2 & 6 \\ 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 6 & 2 \end{pmatrix},$$

and

$$[Q(\mathcal{E}_{12})] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Obviously, $\operatorname{inx} \mathcal{E}_{12} = 7$*, but the sum of all elements of the first column is 9.*

The right eigenvector of $[Q(\mathcal{E})]$ *is* $(1,1,1,1,1,1,1,1,1,1,1,1)^T$.

Our computations show that left eigenvector of $[Q(\mathcal{E}_{12})]$ *which corresponds to the eigenvalue 7 is*

$$(15,\ 15,\ 8,\ 8,\ 10,\ 10,\ 16,\ 16,\ 14,\ 14,\ 11,\ 11).$$

We have the following statement.

Proposition 4.1. *Let Q be a simply laced strongly connected quiver, d be an arbitrary positive real number. There exists a unique right positive eigenvector $\vec{u}$ with the eigenvalue $r = \operatorname{inx} Q$ and $\|\vec{u}\| = d$.*

Proof. By the Frobenius theorem there exists a positive right eigenvector $\vec{x}$ with the eigenvalue r. Let $\|\vec{x}\| = d_1$, then $\|dd_1^{-1}\vec{x}\| = d$. Let $\vec{u}$ and $\vec{v}$ be two positive eigenvectors with the eigenvalue r, and $\|\vec{u}\| = \|\vec{v}\| = d$. By

Proposition 3.1 and Remark 4.2 $\vec{u} = \alpha\vec{v}$. We have $\|\vec{u}\| = \alpha\|\vec{v}\| = \alpha\|\vec{u}\|$. Therefore, $\alpha = 1$ and $\vec{u} = \vec{v}$. □

Now we can establish our main result.

Theorem 4.2. *Let Q_1 and Q_2 be two isomorphic simply laced strongly connected quivers. Then their characteristic polynomials $\chi_{Q_1}(x)$ and $\chi_{Q_2}(x)$ are equal and positive right (left) eigenvectors $\vec{a}$ and $\vec{b}$ with the maximal eigenvalue r such that $\|\vec{a}\| = \|\vec{b}\|$ are equivalent.*

Proof. We have that $[Q_2] = P_\tau^T[Q_1]P_\tau$. Therefore, $\chi_{Q_1}(x) = \chi_{Q_2}(x)$. Let $\vec{a}$ be a right eigenvector of $[Q_2]$ with the eigenvalue r, and $\vec{b}$ be a right eigenvector of $[Q_2]$ with the eigenvalue r. By Proposition 2.1, $\vec{b}$ is the right eigenvector of $[Q_1]$ with the eigenvalue r. We have that $\|\vec{b}_\tau\| = \|\vec{b}\|$ for any τ. Therefore, $\|\vec{b}_\tau\| = \|\vec{a}\|$. Applying Proposition 4.1 we obtain $\vec{a} = \vec{b}_\tau$.

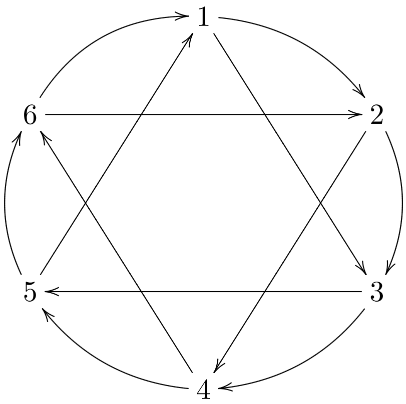
Let Q be a simply laced strongly connected quiver with the adjacency matrix $[Q] = (t_{ij})$. The transpose quiver Q^T is the quiver whose adjacency matrix $[Q^T]$ is equal $[Q]^T$. The quiver Q^T is simply laced and strongly connected if and only if the quiver Q has the same properties. Obviously, $Q_1 \simeq Q_2$ if and only if $Q_1^T = Q_2^T$. If $\vec{b}$ is a left eigenvector of $[Q]^T$, then $\vec{b}^T$ is a right eigenvector of $[Q]$. So, the theorem is proved in the left case. Te right case is proven analogously. □

Applying Theorem 4.2 to the case of Gorenstein quivers we obtain

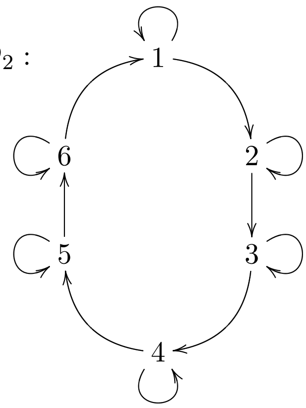
Corollary 4.3. *Let Q_1 and Q_2 be two Gorenstein quivers. If $Q_1 \simeq Q_2$, then $\chi_{Q_1}(x) = \chi_{Q_2}(x)$, $r = inx\, Q_1 = inx\, Q_2$ and right (left) positive eigenvector $\vec{a}$ of $[Q_1]$ with the eigenvalue r and right (left) positive eigenvector $\vec{b}$ of $[Q_2]$ with the same eigenvalue such that $\|\vec{a}\| = \|\vec{b}\|$ are equivalent.*

Example 4.2. *Consider two Gorenstein quivers:*

Q_1 :



Q_2 :



$Q_1 = Q(\mathcal{E}_6)$, *where*

$$\mathcal{E}_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 \end{pmatrix}$$

is the Gorenstein matrix with the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

$Q_2 = Q(T_6)$, *where*

$$T_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 \end{pmatrix}$$

is the Gorenstein matrix with the same permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Obviously, the adjacency matrices of these quivers are

$$[Q_1] = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad [Q_2] = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We have that $\operatorname{inx} Q_1 = \operatorname{inx} Q_2 = 2$. *The left eigenvector* $(1,1,1,1,1,1)$ *of* $[Q_1]$ *is the left eigenvector of* $[Q_2]$. *The right eigenvector* $(1,1,1,1,1,1)^T$ *of* $[Q_1]$ *is the right eigenvector of* $[Q_2]$. *It is easy to see that* $\chi_{Q_1}(x) = (x+1)^2 x(x-2)(x^2+3)$ *and* $\chi_{Q_2}(x) = x(x-2)(x^4 - 4x^3 + \lambda x^2 - 6x + 3)$. *By Theorem 4.2 the quivers* Q_1 *and* Q_2 *are non-isomorphic.*

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