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Robustness of nonuniform dichotomies with different growth rates

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Dedicated to Waldyr Oliva on the occasion of his 80th birthday

Abstract. For nonautonomous linear differential equations $v' = A(t)v$ in a Banach space, we consider general exponential dichotomies that extend the notion of (uniform) exponential dichotomy in various ways. Namely, the new notion allows: stable and unstable behavior with respect to growth rates $e^{c\rho(t)}$ for an arbitrary function $\rho(t)$; nonuniform exponential behavior, causing that any stability or conditional stability may be nonuniform; and different growth rates in the uniform and nonuniform parts of the dichotomy. Our objective is threefold:

1. to show that there is a large class of linear differential equations admitting this general exponential behavior;
2. to provide conditions for the existence of general dichotomies in terms of appropriate Lyapunov exponents;
3. to establish the robustness of the exponential behavior, that is, its persistence under sufficiently small linear perturbations.

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1. Introduction

For nonautonomous linear equations

$$v' = A(t)v, \tag{1}$$

where $A\colon \mathbb{R}_0^+ \to \mathcal{B}(X)$ is a continuous function with values in the space of bounded linear operators in a Banach space X, we consider a very general type of exponential behavior. This generalizes the classical notion of exponential dichotomy in various ways: besides introducing a nonuniform term, causing that any stability or conditional stability may be nonuniform, we consider arbitrary rates that in particular may not be exponential, as well as different growth rates in the uniform and nonuniform parts. This includes for example the classical notion of (uniform) exponential dichotomy, as well as the notions of nonuniform exponential dichotomy and nonuniform polynomial dichotomy (a similar comment applies when the word dichotomy is replaced by the word contraction). In particular, the modifications of the notion of (uniform) exponential dichotomy allow a nonuniform stability or even nonuniform conditional stability with respect to the initial condition, as well as situations when the Lyapunov exponents are all infinity or all zero.

We have three main objectives:

1. to show that in a finite-dimensional space a large class of linear differential equations admit this exponential behavior;
2. to provide natural conditions for the existence of general dichotomies in terms of appropriate Lyapunov exponents;
3. to establish the robustness of the exponential behavior, that is, the persistence of the exponential behavior in the equation

   $$v' = [A(t) + B(t)]v \tag{2}$$

   for any sufficiently small linear perturbation $B(t)$.

We emphasize that when compared to the case of uniform and even nonuniform *exponential* behavior, this creates additional complications, particularly in the case of dichotomies. Namely, besides the existence of expansion and contraction, unlike in the uniform case the stable and unstable directions may approach each other. This means that we need to control the “angle” between the two directions, by estimating the norms of the corresponding projections. The fact that we consider different rates in the uniform and nonuniform parts of the dichotomies is an additional complication that requires a careful control of the perturbations.

The classical notion of (uniform) exponential dichotomy, essentially introduced by Perron in [19], plays an important role in a large part of the theory of differential equations and dynamical systems. In particular, the existence of an exponential dichotomy for equation (1) implies the existence of topological conjugacies and of stable and unstable invariant manifolds for the equation

$$v' = A(t)v + f(t, v)$$

for any sufficiently small nonlinear perturbation f. We refer the reader to the books [8, 12, 13, 22] for details and references. On the other hand, it is also true that the existence of an exponential dichotomy is a stringent condition and it is important to look for more general notions, particularly in view of the applications.

In particular, the notion of nonuniform exponential dichotomy (see [5]) was inspired both on the classical notion of (uniform) exponential dichotomy and on the notion of nonuniformly hyperbolic trajectory introduced by Pesin (see [2]). We emphasize that in comparison to the notion of (uniform) exponential dichotomy, this is a much weaker requirement. For example, in finite-dimensional spaces essentially any linear differential equation with nonzero Lyapunov exponents admits a nonuniform exponential dichotomy (and any linear differential equation with negative Lyapunov exponents admits a nonuniform exponential contraction).

In this paper we consider an even more general type of exponential dichotomy in which the usual exponential behavior is replaced by an arbitrary growth rate. This may correspond for example to situations when the Lyapunov exponents are all infinity or are all zero. More precisely, Barreira and Valls introduced in [6] the notion of ρ-exponential dichotomy. This corresponds to assume that the linear equation (1) may exhibit stable and unstable behaviors with asymptotic rates of the form $e^{c\rho(t)}$, where ρ is some increasing function. The robustness of this general type of exponential dichotomy is also established in [6].

In addition, we consider the case of different growth rates for the uniform and nonuniform parts of the dichotomy, as proposed in [7]. Certainly, if these general dichotomies are supposed to play any role in the theory, one must show that they occur in a sufficiently large class of dynamics. In this paper, besides establishing their occurrence in a large natural class, we also establish their robustness. More precisely, we show in this paper that:

1. there is a large class of equations exhibiting this behavior, and we can provide natural conditions for the existence of general dichotomies in terms of appropriate Lyapunov exponents;
2. the nonuniform part of the asymptotic behavior can be estimated by the so-called regularity coefficient, which is defined in terms of the Lyapunov exponents of equation (1) and its adjoint.

The so-called robustness problem also has a long history. In particular, the problem was discussed by Massera and Schäffer [16], Perron [19], Coppel [11] and in the case of Banach spaces by Dalec′kiĭ and Kreĭn [10]. The continuous dependence of the projections of the exponential dichotomies on the perturbation was obtained by Palmer [18] for ordinary differential equations. For more recent work we refer to [9, 14, 15, 17, 20, 21] and the references therein. We also refer to [4, 6] for the study of robustness of nonuniform exponential dichotomies.

I. NONUNIFORM CONTRACTIONS

We first concentrate on the simpler case of nonuniform contractions, leaving the more elaborate case of nonuniform dichotomies for the second part of the paper. This allows us to present the results and their proofs without some accessory technicalities. After the introduction of some basic notions, we show how appropriate Lyapunov exponents can be used to deduce that a linear differential equation with negative Lyapunov exponents in a finite-dimensional space admits a nonuniform contraction. We also show how the nonuniform part of the contraction can be estimated by the so-called regularity coefficient, which is defined in terms of the Lyapunov exponents of equation (1) and its adjoint. Finally, we establish the robustness of nonuniform contractions, that is, we establish the persistence of the stable behavior under sufficiently small linear perturbations.

2. Basic notions

Let $\mathcal{B}(X)$ be the space of bounded linear operators in a Banach space X. Given a continuous function $A\colon \mathbb{R}_0^+ \to \mathcal{B}(X)$, we denote by $T(t,s)$ the evolution operator associated to equation (1). This is the linear operator such that

$$T(t,s)v(s) = v(t), \quad t,s \geq 0,$$

where $v(t)$ is any solution of equation (1) (we note that any solution is global). Clearly, $T(t,t) = \mathrm{Id}$ and

$$T(t,\tau)T(\tau,s) = T(t,s), \quad t,\tau,s \geq 0.$$

In order to introduce the notion of nonuniform contraction, it is convenient to consider the notion of growth rate. We say that an increasing function $\mu\colon \mathbb{R}_0^+ \to [1,+\infty)$ is a *growth rate* if

$$\mu(0) = 1 \quad \text{and} \quad \lim_{t\to+\infty} \mu(t) = +\infty.$$

Given growth rates μ and ν, we say that equation (1) admits a (μ,ν)-*nonuniform contraction* if there exist constants $\alpha, D > 0$ and $\varepsilon \geq 0$ such that

$$\|T(t,s)\| \le D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu(s)^{\varepsilon}, \quad t \ge s \ge 0. \tag{3}$$

The constant $-\alpha$ plays the role of a Lyapunov exponent, while ε measures the nonuniformity of the contraction (it is also related to the Lyapunov exponents, although in a more subtler manner; see Section 4). For example, if μ and ν are arbitrary differentiable growth rates, and $\alpha > 0$ and $\varepsilon \ge 0$, then the scalar differential equation

$$v' = \left(\frac{-\alpha\mu'(t)}{\mu(t)} + \frac{\varepsilon\nu'(t)}{2\nu(t)}(\cos t - 1) - \frac{\varepsilon}{2}\log\nu(t)\sin t\right)v$$

admits a (μ,ν)-nonuniform contraction with $D = 1$.

When $\mu(t) = \nu(t) = e^{\rho(t)}$, we recover the notion of *ρ-nonuniform exponential contraction*, in which case (3) reduces to

$$\|T(t,s)\| \le De^{-\alpha(\rho(t)-\rho(s))+\varepsilon\rho(s)}, \quad t \ge s \ge 0.$$

When $\mu(t) = \nu(t) = 1 + t$, we recover the notion of *nonuniform polynomial contraction*, in which case (3) reduces to

$$\|T(t,s)\| \le D\left(\frac{1+t}{1+s}\right)^{-\alpha}(1+s)^{\varepsilon}, \quad t \ge s \ge 0.$$

3. **Lyapunov exponents**

We introduce in this section a notion of Lyapunov exponent for linear differential equations in a finite-dimensional space that is well adapted to an arbitrary growth rate. This shall be useful in Sections 4 and 5.

We consider equation (1) for some continuous function $A\colon \mathbb{R}_0^+ \to M_n(\mathbb{R})$ with values in the set $M_n(\mathbb{R})$ of $n \times n$ matrices. Given a growth rate μ, we define a new function $\chi\colon \mathbb{R}^n \to [-\infty, +\infty]$ by

$$\chi(v_0) = \limsup_{t\to+\infty}\frac{\log\|v(t)\|}{\log\mu(t)}, \tag{4}$$

where $v(t)$ is the solution of equation (1) with $v(0) = v_0$ (with the convention that $\log 0 = -\infty$). We say that χ is the *μ-Lyapunov exponent* associated to equation (1).

We first describe a few properties of the function χ. The following statement is an easy consequence of the definition.

Proposition 1. *The function χ is a Lyapunov exponent, that is, the following properties hold:*

1. $\chi(0) = -\infty$*;*

2. $\chi(cv) = \chi(v)$ *for every* $c \in \mathbb{R} \setminus \{0\}$ *and* $v \in \mathbb{R}^n$*;*
3. $\chi(u+v) \le \max\{\chi(u), \chi(v)\}$ *for every* $u, v \in \mathbb{R}^n$*.*

Since χ is a Lyapunov exponent, it follows from the abstract theory of Lyapunov exponents (see [1, Section 1.2]) that:

1. $\chi(u+v) = \max\{\chi(u), \chi(v)\}$ whenever $\chi(u) \ne \chi(v)$;
2. if for some nonzero vectors $u_1, \dots, u_m$ the numbers $\chi(u_1), \dots, \chi(u_m)$ are distinct, then $u_1, \dots, u_m$ are linearly independent;
3. the function χ attains at most $n+1$ distinct values.

In particular, χ can take only finitely many values in $\mathbb{R}^n \setminus \{0\}$, that we denote by $\lambda_1 < \cdots < \lambda_r$ for some integer $r \le n$. We note that in general λ_1 may be $-\infty$ and λ_r may be $+\infty$. For each $i = 1, \dots, r$, we consider the set

$$V_i = \{v \in \mathbb{R}^n : \chi(v) \le \lambda_i\}.$$

It follows from Proposition 1 that V_i is a vector space. The number

$$k_i = \dim V_i - \dim V_{i-1},$$

with the convention that $V_0 = \{0\}$, is called the *multiplicity* of the value λ_i.

In order to introduce the notion of Lyapunov regularity, we first consider the adjoint equation

$$w' = -A(t)^* w, \tag{5}$$

where $A(t)^*$ denotes the transpose of the matrix $A(t)$. Given a growth rate ν, we define the *ν-Lyapunov exponent* $\tilde{\chi} \colon \mathbb{R}^n \to [-\infty, +\infty]$ associated to equation (5) by

$$\tilde{\chi}(w_0) = \limsup_{t \to +\infty} \frac{\log\|w(t)\|}{\log \nu(t)}, \tag{6}$$

where $w(t)$ is the solution of equation (5) with $w(0) = w_0$. Again by the abstract theory of Lyapunov exponents, $\tilde{\chi}$ can take only finitely many values in $\mathbb{R}^n \setminus \{0\}$. We denote them by $\tilde{\lambda}_s < \cdots < \tilde{\lambda}_1$ for some integer $s \le n$. In general $\tilde{\lambda}_s$ may be $-\infty$ and $\tilde{\lambda}_1$ may be $+\infty$.

We define the *regularity coefficient* of the Lyapunov exponents χ and $\tilde{\chi}$ by

$$\gamma = \min \max\{\chi(v_i) + \tilde{\chi}(w_i) : 1 \le i \le n\}, \tag{7}$$

where the minimum is taken over all dual bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ of $\mathbb{R}^n$, that is, all bases such that

$$\langle v_i, w_j \rangle = \delta_{ij} \quad \text{for} \quad i, j = 1, \dots, n$$

(here δ_{ij} is the Kronecker symbol). In order to ensure that the regularity coefficient is well defined, we always assume in the paper that the sums $\lambda_1 + \tilde{\lambda}_1$ and $\lambda_r + \tilde{\lambda}_s$ are well defined.

4. Existence of nonuniform contractions

We show in this section that in a finite-dimensional space a large class of linear differential equations having only negative Lyapunov exponents admits a nonuniform contraction. The more general case of nonuniform dichotomies can be treated in a similar manner, although the presentation would be much more involved, and thus we have chosen not to include it in the paper.

Theorem 1. *Let $A(t)$ be a $n \times n$ matrix varying continuously with $t \geq 0$, and let $\mu \geq \nu$ be growth rates. If $\chi(v) < 0$ for every $v \in \mathbb{R}^n$, then for each sufficiently small $\delta > 0$ equation (1) admits a (μ, ν)-nonuniform contraction with*

$$\alpha = -(\lambda_r + \delta) \quad \textit{and} \quad \varepsilon = \gamma + 2\delta.$$

Moreover, there exists $C > 0$ such that

$$\|T(t,s)^{-1}\| \leq C\left(\frac{\mu(s)}{\mu(t)}\right)^{\lambda_1+\delta} \nu(t)^{\gamma+2\delta}, \quad t \geq s \geq 0. \tag{8}$$

Proof. The argument is based on the proof of Theorem 4 in [3]. We write

$$T(t,s) = X(t)X(s)^{-1},$$

where $X(t)$ is a fundamental solution matrix of the equation $v' = A(t)v$. One can easily verify that the matrix $Y(t) = [X(t)^*]^{-1}$ satisfies

$$Y'(t) = -A(t)^*Y(t).$$

Since $Y(t)$ is invertible for each $t \geq 0$, its columns form a basis of the space of solutions of the equation $w' = -A(t)^*w$, and thus $Y(t)$ is a fundamental solution matrix of this equation. Now let $x_1(t), \ldots, x_n(t)$ be the columns of $X(t)$, and let $y_1(t), \ldots, y_n(t)$ be the columns of $Y(t)$. For each $j = 1, \ldots, n$, we set

$$\alpha_j = \chi(x_j(0)) \quad \text{and} \quad \beta_j = \tilde{\chi}(y_j(0)),$$

where χ and $\tilde{\chi}$ are respectively the Lyapunov exponents in (4) and (6). We also write

$$\rho(t) = \log\mu(t) \quad \text{and} \quad \sigma(t) = \log\nu(t). \tag{9}$$

Given $\delta > 0$ such that $\lambda_r + \delta < 0$, there exists $D > 0$ such that

$$\|x_j(t)\| \leq De^{(\alpha_j+\delta)\rho(t)} \quad \text{and} \quad \|z_j(t)\| \leq De^{(\beta_j+\delta)\sigma(t)}$$

for every $j = 1, \ldots, n$ and $t \geq 0$. Since $X(t)^*Y(t) = \text{Id}$, we have

$$\langle x_i(t), y_j(t)\rangle = \delta_{ij} \text{ for every } i \text{ and } j.$$

Therefore, taking into account that the minimum in (7) can take only finitely many values, eventually rechoosing the matrix $X(t)$ one can always assume that

$$\max\{\alpha_j+\beta_j : j=1,\dots,n\}=\gamma.$$

The entries of the matrix $T(t,s)=X(t)Y(s)^*$ are

$$u_{ik}(t,s)=\sum_{j=1}^n x_{ij}(t)y_{kj}(s),$$

where $x_{ij}(t)$ is the ith coordinate of $x_j(t)$ and where $y_{kj}(s)$ is the kth coordinate of $y_j(s)$. Since $\alpha_j+\delta\le\lambda_r+\delta<0$ and $\rho\ge\sigma$, we obtain

$$\begin{aligned}
|u_{ik}(t,s)| &\le \sum_{j=1}^n |x_{ij}(t)|\cdot|y_{kj}(s)| \le \sum_{j=1}^n \|x_j(t)\|\cdot\|y_j(s)\| \\
&\le \sum_{j=1}^n D^2 e^{(\alpha_j+\delta)\rho(t)+(\beta_j+\delta)\sigma(s)} \\
&\le \sum_{j=1}^n D^2 e^{(\alpha_j+\delta)(\rho(t)-\rho(s))+(\alpha_j+\delta)\rho(s)+(\beta_j+\delta)\sigma(s)} \\
&= \sum_{j=1}^n D^2 e^{(\alpha_j+\delta)(\rho(t)-\rho(s))+(\alpha_j+\beta_j+2\delta)\sigma(s)} \\
&\le nD^2 e^{(\lambda_r+\delta)(\rho(t)-\rho(s))+(\gamma+2\delta)\sigma(s)}.
\end{aligned}$$

Taking a vector $v=\sum_{k=1}^n c_k e_k$, where $e_1,\dots,e_n$ is the canonical basis of $\mathbb{R}^n$, if $\|v\|^2=\sum_{k=1}^n c_k^2=1$, then

$$\begin{aligned}
\|T(t,s)\|^2 &= \left\|\sum_{i=1}^n\sum_{k=1}^n c_k u_{ik}(t,s)e_i\right\|^2 \\
&\le \sum_{i=1}^n\left(\sum_{k=1}^n c_k^2\sum_{k=1}^n u_{ik}(t,s)^2\right) = \sum_{i=1}^n\sum_{k=1}^n u_{ik}(t,s)^2.
\end{aligned}\tag{10}$$

Therefore, writing $D'=n^2D^2$, we conclude that

$$\begin{aligned}
\|T(t,s)\| &\le \left(\sum_{i=1}^n\sum_{k=1}^n u_{ik}(t,s)^2\right)^{1/2} \\
&\le D' e^{(\lambda_r+\delta)(\rho(t)-\sigma(s))+(\gamma+2\delta)\sigma(s)}.
\end{aligned}$$

This shows that equation (1) admits a (μ, ν)-nonuniform contraction with $\alpha = -(\lambda_r + \delta)$ and $\varepsilon = \gamma + 2\delta$.

Now we obtain the bound in (8). We have $T(t,s)^{-1} = X(s)Y(t)^*$, and thus the entries of this matrix are

$$u_{ik}(s,t) = \sum_{j=1}^{n} x_{ij}(s) y_{kj}(t).$$

Since $\rho \geq \sigma$, we obtain

$$\begin{aligned}
|u_{ik}(s,t)| &\leq \sum_{j=1}^{n} |x_{ij}(s)| \cdot |y_{kj}(t)| \leq \sum_{j=1}^{n} \|x_j(s)\| \cdot \|y_j(t)\| \\
&\leq \sum_{j=1}^{n} D^2 e^{(\alpha_j+\delta)\rho(s)+(\beta_j+\delta)\sigma(t)} \\
&= \sum_{j=1}^{n} D^2 e^{(\alpha_j+\delta)(\rho(s)-\rho(t))+(\alpha_j+\delta)\rho(t)+(\beta_j+\delta)\sigma(t)} \\
&\leq nD^2 e^{(\lambda_1+\delta)(\rho(s)-\rho(t))+(\gamma+2\delta)\sigma(t)}.
\end{aligned}$$

Proceeding as in (10), one can show that this yields the estimate in (8). $\square$

5. Bounds for the regularity coefficient

We obtain in this section lower and upper bounds for the regularity coefficient γ (we recall that it was introduced only in the finite-dimensional setting). This is particularly interesting in view of Theorem 1, which tells us that the nonuniform part of a nonuniform contraction is essentially measured by the regularity coefficient. The proofs in this section are appropriate modifications of arguments in [3].

We first obtain a lower bound for the regularity coefficient. For simplicity of the notation we continue to use the functions ρ and σ in (9).

Theorem 2. *For the regularity coefficient γ in (7), we have*

$$\gamma \geq \frac{1}{n}\left(\limsup_{t\to+\infty} \frac{1}{\rho(t)} \int_0^t \operatorname{tr} A(\tau)\, d\tau - \liminf_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t \operatorname{tr} A(\tau)\, d\tau\right).$$

Proof. Let $v_1, \ldots, v_n$ be a basis of $\mathbb{R}^n$, and for each i let $v_i(t)$ be the solution of equation (1) with $v_i(0) = v_i$. The matrix $X(t)$ whose columns are the vectors $v_1(t), \ldots, v_n(t)$ is nonsingular and satisfies

$$\frac{\det X(t)}{\det X(0)} = \exp \int_0^t \operatorname{tr} A(\tau)\, d\tau$$

for every $t \geq 0$. Furthermore,

$$|\det X(t)| \leq \prod_{i=1}^{n} \|v_i(t)\|,$$

and thus,

$$\limsup_{t\to+\infty} \frac{1}{\rho(t)} \int_0^t \operatorname{tr} A(\tau)\, d\tau \leq \sum_{i=1}^{n} \chi(v_i). \tag{11}$$

Now let $w_1, \ldots, w_n$ be another basis of $\mathbb{R}^n$. By the former argument, we have

$$\begin{aligned}\liminf_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t \operatorname{tr} A(\tau)\, d\tau &= -\limsup_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t \operatorname{tr}(-A(\tau)^*)\, d\tau \\ &\geq -\sum_{i=1}^{n} \tilde{\chi}(w_i).\end{aligned} \tag{12}$$

Adding the inequalities in (11) and (12) yields

$$\limsup_{t\to+\infty} \frac{1}{\rho(t)} \int_0^t \operatorname{tr} A(\tau)\, d\tau - \liminf_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t \operatorname{tr} A(\tau)\, d\tau \leq \sum_{i=1}^{n} (\chi(v_i) + \tilde{\chi}(w_i)).$$

If the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are dual, and the minimum in (7) is attained at this pair, that is,

$$\gamma = \max\{\chi(v_i) + \tilde{\chi}(w_i) : 1 \leq i \leq n\},$$

then

$$\sum_{i=1}^{n} (\chi(v_i) + \tilde{\chi}(w_i)) \leq n \max\{\chi(v_i) + \tilde{\chi}(w_i) : 1 \leq i \leq n\} = n\, \gamma.$$

Together with (12) this establishes the desired result. □

For example, when $n = 1$, that is, for the scalar equation $v' = a(t)v$, we obtain

$$\gamma \geq \limsup_{t\to+\infty} \frac{1}{\rho(t)} \int_0^t a(\tau)\, d\tau - \liminf_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t a(\tau)\, d\tau. \tag{13}$$

To obtain an upper bound we assume that the matrices $A(t)$ are upper triangular. It follows for example from Lemma 1.3.3 in [1] that there is no loss of generality in making this assumption. We also assume that

$$\lim_{t\to+\infty} \frac{\log t}{\rho(t)} = 0, \quad \lim_{t\to+\infty} \frac{\log t}{\sigma(t)} = 0, \tag{14}$$

and

$$\limsup_{t\to+\infty} \frac{1}{\rho(t)} \log^+\|A(t)\| = 0, \quad \limsup_{t\to+\infty} \frac{1}{\sigma(t)} \log^+\|A(t)\| = 0, \tag{15}$$

where $\log^+ x = \max\{\log x, 0\}$, again with ρ and σ as in (9). Finally, we consider the numbers

$$\overline{c}_k = \limsup_{t\to+\infty} \frac{1}{\rho(t)} \int_0^t a_k(\tau)\, d\tau \tag{16}$$

and

$$\underline{c}_k = \liminf_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t a_k(\tau)\, d\tau, \tag{17}$$

where $a_1(t), \ldots, a_n(t)$ are the entries in the diagonal of $A(t)$.

Theorem 3. *Assume that $A(t)$ is upper triangular for every $t \geq 0$, and that conditions* (14) *and* (15) *hold for some differentiable functions $\rho \geq \sigma$. If $\underline{c}_k, \overline{c}_k \leq 0$ for $k = 1, \ldots, n$, then*

$$\gamma \leq \sum_{m=1}^{n} (\overline{c}_m - \underline{c}_m).$$

Proof. We first establish two auxiliary results. Set

$$c_{ij} = \overline{c}_j - \underline{c}_i + \sum_{m=i+1}^{j-1} (\overline{c}_m - \underline{c}_m), \tag{18}$$

and take

$$h_{ij} = \begin{cases} 0 & \text{if } c_{ij} \geq 0, \\ +\infty & \text{if } c_{ij} < 0. \end{cases} \tag{19}$$

We denote by $a_{ij}(t)$ the entries of the matrix $A(t)$. For each $i = 1, \ldots, n$ and $t \geq 0$, set

$$v_{ij}(t) = \begin{cases} 0 & \text{if } i > j, \\ e^{\int_0^t a_{ii}(\tau)\, d\tau} & \text{if } i = j, \\ \int_{h_{ij}}^t \sum_{k=i+1}^{j} a_{ik}(s) v_{kj}(s) e^{\int_s^t a_{ii}(\tau)\, d\tau}\, ds & \text{if } i < j. \end{cases} \tag{20}$$

One cay easily verify that the columns of the matrix $V(t) = (v_{ij}(t))$ form a basis of the space of solutions of equation (1). These columns are

$$v_j(t) = (v_{1j}(t), \ldots, v_{nj}(t)) \tag{21}$$

for $j = 1, \ldots, n$. Given $i, j = 1, \ldots, n$, we also set

$$\lambda_{ij} = \limsup_{t\to+\infty} \frac{1}{\rho(t)} \log |v_{ij}(t)|.$$

Lemma 1. *For each $i, j = 1, \dots, n$, we have*

$$\lambda_{ij} \le \overline{c}_j + \sum_{m=i}^{j-1} (\overline{c}_m - \underline{c}_m).$$

Proof of the lemma. Clearly,

$$\lambda_{ii} = \overline{c}_i \quad \text{and} \quad i = 1, \dots, n.$$

Now we proceed by backward induction on i. Namely, for a given $i < n$, we assume that

$$\lambda_{kj} \le \overline{c}_j + \sum_{m=k}^{j-1} (\overline{c}_m - \underline{c}_m) \quad \text{whenever} \quad i+1 \le k \le j. \tag{22}$$

We shall prove that for $j \ge i+1$,

$$\lambda_{ij} \le \overline{c}_j + \sum_{m=i}^{j-1} (\overline{c}_m - \underline{c}_m). \tag{23}$$

By (15), (17), and (22), for each $\varepsilon > 0$ sufficiently small there exists $D > 0$ such that $|a_{ik}(t)| \le D e^{\varepsilon \rho(t)}$,

$$e^{-\int_0^t a_{ii}(\tau)\, d\tau} \le D e^{(-\underline{c}_i + \varepsilon)\sigma(t)} \le D e^{(-\underline{c}_i + \varepsilon)\rho(t)},$$

and

$$|v_{kj}(t)| \le D e^{[\overline{c}_j + \sum_{m=k}^{j-1} (\overline{c}_m - \underline{c}_m) + \varepsilon]\rho(t)}$$

for every $t \ge 0$ and $i+1 \le k \le j$. Therefore,

$$\begin{aligned}
\lambda_{ij} &\le \limsup_{t\to+\infty} \frac{1}{\rho(t)} \log \left(e^{\int_0^t a_{ii}(\tau)\, d\tau} \left| \int_{h_{ij}}^t \sum_{k=i+1}^{j} |a_{ik}(s) v_{kj}(s)| e^{-\int_0^s a_{ii}(\tau)\, d\tau}\, ds \right| \right) \\
&\le \overline{c}_i + \limsup_{t\to+\infty} \frac{1}{\rho(t)} \log \left| \int_{h_{ij}}^t D^3 \sum_{k=i+1}^{j} e^{[\overline{c}_j + \sum_{m=k}^{j-1} (\overline{c}_m - \underline{c}_m) - \underline{c}_i + 3\varepsilon]\rho(s)}\, ds \right| \\
&\le \overline{c}_i + \limsup_{t\to+\infty} \frac{1}{\rho(t)} \log \left| \int_{h_{ij}}^t D^3 n e^{[\overline{c}_j + \sum_{m=i+1}^{j-1} (\overline{c}_m - \underline{c}_m) - \underline{c}_i + 3\varepsilon]\rho(s)}\, ds \right| \\
&= \overline{c}_i + \limsup_{t\to+\infty} \frac{1}{\rho(t)} \log \left| \int_{h_{ij}}^t e^{(c_{ij} + 3\varepsilon)\rho(s)}\, ds \right|.
\end{aligned} \tag{24}$$

Now we consider two cases.

Case $c_{ij} \geq 0$. Since ρ is increasing, for each $t \geq 0$ we have

$$\int_0^t e^{(c_{ij}+3\varepsilon)\rho(s)}\, ds \leq t e^{(c_{ij}+3\varepsilon)\rho(t)}. \tag{25}$$

Case $c_{ij} < 0$. Take $\varepsilon > 0$ sufficiently small so that $a = c_{ij} + 3\varepsilon < 0$. It follows from (14) that $\rho(t) \to +\infty$ when $t \to +\infty$, and thus also that $t\rho'(t) \to +\infty$ when $t \to +\infty$. Therefore, for each $c > 0$ there exists $t_0 > 0$ such that $t\rho'(t) > c$ for every $t > t_0$. For any such t, we obtain

$$\begin{aligned} c\int_t^{+\infty} e^{a\rho(s)}\, ds &\leq \int_t^{+\infty} s\rho'(s)e^{a\rho(s)}\, ds \\ &= \frac{s}{a}e^{a\rho(s)}\Big|_t^{+\infty} - \frac{1}{a}\int_t^{+\infty} e^{a\rho(s)}\, ds. \end{aligned}$$

By condition (14), we have

$$\log(se^{a\rho(s)}) = \log s + a\rho(s) = \rho(s)\left(\frac{\log s}{\rho(s)} + a\right) \to -\infty$$

when $s \to +\infty$, and thus,

$$\left(c + \frac{1}{a}\right)\int_t^{+\infty} e^{a\rho(s)}\, ds \leq \frac{t}{|a|}e^{a\rho(t)}.$$

Taking c sufficiently large and $t > t_0$, we find that

$$\int_t^{+\infty} e^{(c_{ij}+3\varepsilon)\rho(s)}\, ds \leq t e^{(c_{ij}+3\varepsilon)\rho(t)}. \tag{26}$$

By (25) and (26), in both cases we obtain

$$\limsup_{t\to+\infty} \frac{1}{\rho(t)} \log\left|\int_{h_{ij}}^t e^{(c_{ij}+3\varepsilon)\rho(s)}\, ds\right| \leq c_{ij} + 3\varepsilon.$$

Hence, it follows from (24) that

$$\lambda_{ij} \leq \overline{c}_i + c_{ij} + 3\varepsilon = \overline{c}_j + \sum_{m=i}^{j-1}(\overline{c}_m - \underline{c}_m) + 3\varepsilon.$$

Since ε is arbitrary, we conclude that (23) holds for every $j \geq i + 1$. This completes the proof of the lemma. $\square$

Now we consider the adjoint equation in (5). For each $i,j=1,\dots,n$ and $t\ge 0$, set

$$w_{ij}(t)=\begin{cases}0 & \text{if } i<j,\\ e^{-\int_0^t a_{ii}(\tau)\,d\tau} & \text{if } i=j,\\ -\int_{h_{ji}}^t \sum_{k=j}^{i-1} a_{ki}(s)w_{kj}(s)e^{-\int_s^t a_{ii}(\tau)\,d\tau}\,ds & \text{if } i>j.\end{cases} \tag{27}$$

with the same constants as in (19). One can easily verify that the columns of the matrix $W(t)=(w_{ij}(t))$ form a basis of the space of solutions of equation (5). These columns are

$$w_j(t)=(w_{1j}(t),\dots,w_{nj}(t)) \tag{28}$$

for $j=1,\dots,n$. Given $i,j=1,\dots,n$, we set

$$\tilde\lambda_{ij}=\limsup_{t\to+\infty}\frac{1}{\sigma(t)}\log|w_{ij}(t)|.$$

Lemma 2. *For each $i,j=1,\dots,n$, we have*

$$\tilde\lambda_{ij}\le -\underline{c}_j+\sum_{m=j+1}^{i}(\overline{c}_m-\underline{c}_m). \tag{29}$$

Proof of the lemma. We proceed in a similar manner to that in the proof of Lemma 1. Clearly,

$$\tilde\lambda_{jj}=-\underline{c}_j \quad\text{and}\quad j=1,\dots,n.$$

Now we proceed by induction on i. Namely, given $i>1$, we assume that

$$\tilde\lambda_{kj}\le -\underline{c}_j+\sum_{m=j+1}^{k}(\overline{c}_m-\underline{c}_m) \text{ whenever } \quad j\le k\le i-1. \tag{30}$$

We shall prove that for $j\le i-1$,

$$\tilde\lambda_{ij}\le -\underline{c}_j+\sum_{m=j+1}^{i}(\overline{c}_m-\underline{c}_m).$$

It follows from (15), (16), and (30) that given $\varepsilon>0$ sufficiently small there exists $D>0$ such that $|a_{ki}(t)|\le De^{\varepsilon\sigma(t)}$,

$$e^{\int_0^t a_{ii}(\tau)\,d\tau}\le De^{(\overline{c}_i+\varepsilon)\rho(t)}\le De^{(\overline{c}_i+\varepsilon)\sigma(t)},$$

and

$$|w_{kj}(t)|\le De^{[-\underline{c}_j+\sum_{m=j+1}^{k}(\overline{c}_m-\underline{c}_m)+\varepsilon]\sigma(t)}$$

for every $t \ge 0$ and $j \le k \le i-1$. Therefore,

$$\begin{aligned}\tilde{\lambda}_{ij} &\le \limsup_{t\to+\infty} \frac{1}{\sigma(t)} \log\left(e^{-\int_0^t a_{ii}(\tau)\,d\tau}\left|\int_{h_{ji}}^t \sum_{k=j}^{i-1} |a_{ki}(s)w_{kj}(s)| e^{\int_0^s a_{ii}(\tau)\,d\tau}\,ds\right|\right)\\
&\le -\underline{c}_i + \limsup_{t\to+\infty} \frac{1}{\sigma(t)} \log\left|\int_{h_{ji}}^t D^3 \sum_{k=j}^{i-1} e^{[-\underline{c}_j+\sum_{m=j+1}^k(\overline{c}_m-\underline{c}_m)+\overline{c}_i+3\varepsilon]\sigma(s)}\,ds\right|\\
&\le -\underline{c}_i + \limsup_{t\to+\infty} \frac{1}{\sigma(t)} \log\left|\int_{h_{ji}}^t e^{(c_{ji}+3\varepsilon)\sigma(s)}\,ds\right|.\end{aligned}$$

Proceeding as in the proof of Lemma 1, we obtain

$$\limsup_{t\to+\infty} \frac{1}{\sigma(t)} \log\left|\int_{h_{ji}}^t e^{(c_{ji}+3\varepsilon)\sigma(s)}\,ds\right| \le c_{ji} + 3\varepsilon.$$

Hence,

$$\tilde{\lambda}_{ij} \le -\underline{c}_i + c_{ji} + 3\varepsilon = -\underline{c}_j + \sum_{m=j+1}^{i} (\overline{c}_m - \underline{c}_m) + 3\varepsilon,$$

and the arbitrariness of ε yields (29) for every $j \le i-1$. This completes the proof of the lemma. $\square$

We proceed with the proof of the theorem. Set $v_j = v_j(0)$ and $w_j = w_j(0)$ for each j, with the vectors $v_j(t)$ and $w_j(t)$ as in (21) and (28). By Lemmas 1 and 2 we have

$$\chi(v_j) = \max\{\lambda_{ij} \colon i = 1,\dots,n\} \le \overline{c}_j + \sum_{m=1}^{j-1}(\overline{c}_m - \underline{c}_m),$$

and

$$\tilde{\chi}(w_j) = \max\{\tilde{\lambda}_{ij} \colon i = 1,\dots,n\} \le -\underline{c}_j + \sum_{m=j+1}^{n}(\overline{c}_m - \underline{c}_m).$$

Therefore,

$$\chi(v_j) + \tilde{\chi}(w_j) \le \sum_{m=1}^{n}(\overline{c}_m - \underline{c}_m) \tag{31}$$

for every $i = 1,\dots,n$. By the definition of the regularity coefficient γ, to complete the proof it suffices to show that the bases $v_1,\dots,v_n$ and $w_1,\dots,w_n$ are dual. Since

$$\begin{aligned}\frac{d}{dt}\langle v_i(t), w_j(t)\rangle &= \langle A(t)v_i(t), w_j(t)\rangle + \langle v_i(t), -A(t)^* w_j(t)\rangle\\
&= \langle A(t)v_i(t), w_j(t)\rangle - \langle A(t)v_i(t), w_j(t)\rangle = 0,\end{aligned}$$

we have

$$\langle v_i(t), w_j(t)\rangle = \langle v_i, w_j\rangle \quad \text{for every} \quad t \ge 0.$$

Clearly, $\langle v_i, w_j\rangle = 0$ for every $i < j$, and

$$\langle v_i, w_i\rangle = v_{ii}(0)w_{ii}(0) = 1$$

for $i = 1, \dots, n$. Now we fix $i > j$ and $t \ge 0$. We have

$$\begin{aligned}\langle v_i(t), w_j(t)\rangle &= \sum_{k=j}^{i} v_{ki}(t)w_{kj}(t) \\ &= v_{ji}(t)w_{jj}(t) + v_{ii}(t)w_{ij}(t) + \sum_{k=j+1}^{i-1} v_{ki}(t)w_{kj}(t).\end{aligned} \tag{32}$$

Moreover, by (18),

$$c_{ji} = c_{ki} + c_{jk} \tag{33}$$

for $k = j+1, \dots, i-1$. We consider two cases:

1. If $c_{ji} \ge 0$, then $h_{ji} = 0$ (see (19)). By (33), for every k such that $j+1 \le k \le i-1$ we have either $c_{ki} \ge 0$ or $c_{jk} \ge 0$. Therefore, by (19), we have $h_{ki} = 0$ or $h_{jk} = 0$, and thus either $v_{ki}(0) = 0$ or $w_{kj}(0) = 0$ (by direct substitution of $t = 0$ in (20) and (27)). Furthermore, again since $h_{ji} = 0$, it follows from (20) and (27) that $v_{ji}(0) = w_{ij}(0) = 0$. Hence, evaluating (32) at $t = 0$ we find that all terms in the last sum are zero, and thus $\langle v_i, w_j\rangle = 0$.
2. If $c_{ji} < 0$, then $h_{ji} = +\infty$ (see (19)). By (33) and (19), for every k such that $j+1 \le k \le i-1$ we have either $h_{ki} = +\infty$ or $h_{jk} = +\infty$, and thus $v_{ki}(+\infty) = 0$ or $w_{kj}(+\infty) = 0$. Hence, taking the limit in (32) as $t \to +\infty$ we find that all terms in the last sum are zero, and thus $\langle v_i, w_j\rangle = 0$.

The theorem follows now from (31) and the definition of γ. $\square$

For example, for the scalar equation $v' = a(t)v$, we obtain

$$\gamma \le \limsup_{t\to+\infty} \frac{1}{\rho(t)} \int_0^t a(\tau)\,d\tau - \liminf_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t a(\tau)\,d\tau,$$

which together with (13) yields the formula

$$\gamma = \limsup_{t\to+\infty} \frac{1}{\rho(t)} \int_0^t a(\tau)\,d\tau - \liminf_{t\to+\infty} \frac{1}{\sigma(t)} \int_0^t a(\tau)\,d\tau$$

when $n = 1$ (under the assumptions (14) and (15)).

6. Robustness of nonuniform contractions

Now we return to the infinite-dimensional setting, and we establish the robustness of a nonuniform contraction under sufficiently small linear perturbations. Namely, we consider the perturbed equation (2), for some continuous function $B\colon \mathbb{R}_0^+ \to \mathcal{B}(X)$.

Theorem 4. *Let $A, B\colon \mathbb{R}_0^+ \to \mathcal{B}(X)$ be continuous functions such that equation* (1) *admits a (μ,ν)-nonuniform contraction. If*

$$\|B(t)\| \le \delta \frac{\mu'(t)}{\mu(t)} \nu(t)^{-\varepsilon}, \quad t \ge 0 \tag{34}$$

for some $\delta < \alpha/D$, then equation (2) *admits a (μ,ν)-nonuniform contraction and*

$$\|\hat{T}(t,s)\| \le D \left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha+\delta D} \nu(s)^{\varepsilon}, \quad t \ge s \ge 0, \tag{35}$$

where $\hat{T}(t,s)$ is the evolution operator associated to equation (2).

Proof. We follow arguments in [6]. We consider the set

$$J = \{(t,s) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : t \ge s\},$$

and the Banach space

$$\mathcal{C} = \big\{U\colon J \to \mathcal{B}(X) : U \text{ is continuous and } \|U\| < \infty\big\}$$

with the norm

$$\|U\| = \sup\big\{\|U(t,s)\|\nu(s)^{-\varepsilon} : (t,s) \in J\big\}.$$

Now we define an operator L in the space $\mathcal{C}$ by

$$(LU)(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)\,d\tau$$

for each $U \in \mathcal{C}$. Since

$$\begin{aligned}
\|(LU)(t,s)\| &\le \|T(t,s)\| + \int_s^t \|T(t,\tau)\| \cdot \|B(\tau)\| \cdot \|U(\tau,s)\|\,d\tau \\
&\le D \left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} \nu(s)^{\varepsilon} + \delta D \|U\| \nu(s)^{\varepsilon} \int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \frac{\mu'(\tau)}{\mu(\tau)}\,d\tau,
\end{aligned}$$

We obtain

$$\begin{aligned}\|(LU)\| &\leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} + \delta D\|U\|\int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}d\tau \\ &= D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha} + \frac{\delta D}{\alpha}\|U\|\left\{1-\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\right\} \leq D + \frac{\delta D}{\alpha}\|U\|,\end{aligned}$$

and thus the operator $L\colon \mathcal{C} \to \mathcal{C}$ is well defined. One can show in a similar manner that

$$\|LU_1 - LU_2\| \leq \frac{\delta D}{\alpha}\|U_1 - U_2\|.$$

Since $\delta < \alpha/D$, the operator L is a contraction. Hence there exists a unique function $U \in \mathcal{C}$ such that $LU = U$, thus satisfying

$$U(t,s) = T(t,s) + \int_s^t T(t,\tau)B(\tau)U(\tau,s)d\tau \tag{36}$$

for every $t \geq s \geq 0$. By the variation of parameters formula, we know that $U(t,s) = \hat{T}(t,s)$. Now we set $\phi(t) = \|\hat{T}(t,s)\|$. Since $U \in \mathcal{C}$, the continuous function ϕ is bounded. Moreover,

$$\begin{aligned}\phi(t) &\leq \|T(t,s)\| + \int_s^t \|T(t,\tau)\| \cdot \|B(\tau)\| \cdot \phi(\tau)\,d\tau \\ &\leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu(s)^\varepsilon + \delta D\int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}\phi(\tau)\,d\tau\end{aligned}$$

for every $t \geq s \geq 0$.

Lemma 3. *We have*

$$\phi(t) \leq D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha+\delta D}\nu(s)^\varepsilon, \qquad t \geq s \geq 0.$$

Proof of the lemma. We specify a continuous function Φ by requiring that it satisfies the integral equation

$$\Phi(t) = D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu(s)^\varepsilon + \delta D\int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}\Phi(\tau)d\tau$$

for every $t \geq s \geq 0$. One can verify that Φ satisfies the differential equation

$$\Phi'(t) = (-\alpha + \delta D)\frac{\mu'(t)}{\mu(t)}\Phi(t),$$

and that $\Phi(s) = D\nu(s)^\varepsilon$. Therefore,

$$\Phi(t) = D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha+\delta D}\nu(s)^\varepsilon.$$

Now let $z(t) = \phi(t) - \Phi(t)$. Then

$$z(t) \leq \delta D \int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \frac{\mu'(\tau)}{\mu(\tau)} z(\tau) d\tau, \quad t \geq s. \tag{37}$$

Since ϕ and Φ are bounded functions, we have

$$z = \sup_{t \geq s} z(t) < +\infty.$$

It follows from (37) that

$$\begin{aligned} z &\leq \delta D \sup_{t \geq s} \int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \frac{\mu'(\tau)}{\mu(\tau)} z(\tau) d\tau \\ &\leq \frac{\delta D}{\alpha} z \left\{1 - \left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\right\} \leq \frac{\delta D}{\alpha} z. \end{aligned}$$

Since $\delta < \alpha/D$, we have $z \leq 0$. Therefore, we conclude that $\phi(t) \leq \Phi(t)$ for every $t \geq s$. □

The lemma yields precisely inequality (35), and the proof of the theorem is complete. □

Under the hypotheses of Theorem 4, it follows from (36) that

$$\begin{aligned} \|\hat{T}(t,s) - T(t,s)\| &\leq \int_s^t \|T(t,\tau)\| \cdot \|B(\tau)\| \cdot \|\hat{T}(\tau,s)\| d\tau \\ &\leq \delta D^2 \int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \left(\frac{\mu(\tau)}{\mu(s)}\right)^{-\alpha+\delta D} \nu(s)^\varepsilon \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &= D\nu(s)^\varepsilon \left\{\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha+\delta D} - \left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\right\} \\ &\leq D\nu(s)^\varepsilon \left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha+\delta D}. \end{aligned}$$

II. NONUNIFORM DICHOTOMIES

We consider in this second part a corresponding notion of nonuniform dichotomy. It generalizes the usual notion of exponential dichotomy in several ways: besides introducing a nonuniform term, causing that any conditional stability may be nonuniform, we consider rates that may not be exponential as well as different rates in the uniform and nonuniform parts. After introducing some basic notions, our main aim is to establish the robustness of nonuniform dichotomies. When compared to the case of contractions, this creates substantial complications. Namely, besides the existence of expansion and contraction, the stable and unstable directions may approach each other. This means that we need to control the "angle" between the two directions by estimating the norms of the corresponding projections.

7. Basic notions

We first introduce the notion of nonuniform dichotomy. Given growth rates μ and ν, we say that equation (1) admits a (μ,ν)-*nonuniform dichotomy* if there exist projections $P(t)\colon X\to X$ for each $t>0$ satisfying

$$T(t,s)P(s)=P(t)T(t,s),\quad t\ge s, \tag{38}$$

and there exist constants $\alpha,\beta,D>0$ and $\varepsilon\ge 0$ such that

$$\|T(t,s)P(s)\|\le D\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu(s)^{\varepsilon},\quad t\ge s, \tag{39}$$

and

$$\|T(t,s)Q(s)\|\le D\left(\frac{\mu(s)}{\mu(t)}\right)^{-\beta}\nu(s)^{\varepsilon},\quad s\ge t, \tag{40}$$

where $Q(t)=\mathrm{Id}-P(t)$ for each $t\ge 0$. Identity (38) corresponds to the invariance of the "stable" and "unstable" spaces,

$$E(t)=P(t)(X)\quad\text{and}\quad F(t)=Q(t)(X),$$

that is,

$$T(t,s)E(s)=E(t)\quad\text{and}\quad T(t,s)F(s)=F(t)$$

for every $t,s\ge 0$. When $P(t)=\mathrm{Id}$ for every $t\ge 0$ we recover the notion of (μ,ν)-nonuniform contraction introduced in Section 2.

8. Robustness of nonuniform dichotomies I

We start in this section the study of the robustness of nonuniform dichotomies by establishing a partial result in which we only control the expansion and contraction. More precisely, here we do not consider the norms of the projections determined by the new stable and unstable directions. That problem is addressed in Section 9 under slightly stronger hypotheses.

The following is our partial robustness result. We denote the evolution operator associated to equation (2) by $\hat{T}(t,s)$.

Theorem 5. *Let $A, B\colon \mathbb{R}_0^+ \to \mathcal{B}(X)$ be continuous functions such that equation* (1) *admits a (μ,ν)-nonuniform dichotomy with $\varepsilon < \min\{\alpha,\beta\}$. If $B(t)$ satisfies* (34) *with δ sufficiently small, then there exist projections $\hat{P}(t)$ and $\hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$ such that*

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{P}(s)\| \leq \tilde{D}\left(\frac{\mu(t)}{\mu(s)}\right)^{-\tilde{\alpha}}\nu(s)^{\varepsilon}, \quad t \geq s, \tag{41}$$

and

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{Q}(s)\| \leq \tilde{D}\left(\frac{\mu(s)}{\mu(t)}\right)^{-\tilde{\alpha}}\nu(s)^{\varepsilon}, \quad s \geq t, \tag{42}$$

where $\tilde{D} = \max\{\tilde{D}_1, \tilde{D}_2\}$, with $\tilde{\alpha}, \tilde{D}_1$ and $\tilde{D}_2$ as in (45) *and* (53).

Proof. We shall always take

$$\delta < \min\left\{\frac{\alpha+\beta}{4D}, \frac{\alpha\beta}{2D(\alpha+\beta)}, \frac{\tilde{\alpha}+\beta}{D}, \frac{\tilde{\alpha}+\alpha}{D}\right\}. \tag{43}$$

The following lemmas can be obtained by repeating arguments in [6], and thus their proofs are omitted. We use the notation introduced in the proof of Theorem 4.

Lemma 4. *There exists a unique function $U \in \mathcal{C}$ such that for each $(t,s) \in J$ we have*

$$\begin{aligned} U(t,s) = T(t,s)P(s) &+ \int_s^t T(t,\tau)P(\tau)B(\tau)U(\tau,s)d\tau \\ &- \int_t^\infty T(t,\tau)Q(\tau)B(\tau)U(\tau,s)d\tau. \end{aligned}$$

Moreover:

1. *$t \mapsto U(t,s)\xi, t \geq s$ is a solution of equation* (2) *for each $\xi \in X$;*
2. *$U(t,\tau)U(\tau,s) = U(t,s)$ for each $t \geq \tau \geq s \geq 0$.*

Using the function U in Lemma 4, we define linear operators

$$\hat{P}(t) = \hat{T}(t,0)U(0,0)\hat{T}(0,t), \quad \hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$$

for each $t \geq 0$. One can easily verify that $\hat{P}(t)$ is a projection for each $t \geq 0$, and that

$$\hat{T}(t,s)\hat{P}(s) = \hat{P}(t)\hat{T}(t,s), \quad t \geq s. \tag{44}$$

Lemma 5. *The function* $[s,+\infty) \ni t \to \hat{P}(t)\hat{T}(t,s)$ *is bounded, and for each* $t \geq s \geq 0$ *we have*

$$\hat{P}(t)\hat{T}(t,s) = T(t,s)P(s)\hat{P}(s) + \int_s^t T(t,\tau)P(\tau)B(\tau)\hat{P}(\tau)\hat{T}(\tau,s)d\tau - \int_t^\infty T(t,\tau)Q(\tau)B(\tau)\hat{P}(\tau)\hat{T}(\tau,s)d\tau.$$

Now we establish the inequalities (41) and (42). Let $\xi \in X$. We set

$$\varphi(t) = \|\hat{P}(t)\hat{T}(t,s)\xi\| \quad \text{for} \quad t \geq s,$$

and $\gamma = \|\hat{P}(s)\xi\|$. By Lemma 5 and (39)–(40), the function φ is bounded, and satisfies

$$\begin{aligned}\varphi(t) &\leq \|T(t,s)P(s)\|\gamma + \int_s^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \varphi(\tau)d\tau \\ &\quad + \int_t^\infty \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\|\varphi(\tau)d\tau \\ &\leq D\gamma\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu(s)^\varepsilon + \delta D\int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}\varphi(\tau)d\tau \\ &\quad + \delta D\int_t^\infty \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta}\frac{\mu'(\tau)}{\mu(\tau)}\varphi(\tau)d\tau.\end{aligned}$$

Lemma 6. *We have*

$$\varphi(t) \leq \tilde{D}_1\gamma\left(\frac{\mu(t)}{\mu(s)}\right)^{-\tilde{\alpha}}\nu(s)^\varepsilon, \qquad t \geq s \geq 0,$$

with the positive constants

$$\tilde{\alpha} = \frac{(\alpha-\beta) + \sqrt{(\alpha+\beta)^2 - 4\delta D(\alpha+\beta)}}{2}, \quad \tilde{D}_1 = \frac{D}{1-\delta D/(\beta+\tilde{\alpha})}. \tag{45}$$

Proof of the lemma. We specify a bounded continuous function Γ by requiring that it satisfies the integral equation

$$\begin{aligned}\Gamma(t) &= D\gamma\left(\frac{\mu(t)}{\mu(s)}\right)^{-\alpha}\nu(s)^\varepsilon + \delta D\int_s^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha}\frac{\mu'(\tau)}{\mu(\tau)}\Gamma(\tau)d\tau \\ &\quad + \delta D\int_t^\infty \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta}\frac{\mu'(\tau)}{\mu(\tau)}\Gamma(\tau)d\tau\end{aligned} \tag{46}$$

for every $t \geq s$. One can verify that

$$\Gamma'(t) = -\alpha\frac{\mu'(t)}{\mu(t)}\Gamma(t) + (\alpha+\beta)\delta D\frac{\mu'(t)}{\mu(t)}\int_t^\infty \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta}\frac{\mu'(\tau)}{\mu(\tau)}\Gamma(\tau)d\tau, \tag{47}$$

and that

$$\begin{aligned}\Gamma''(t) = \Gamma(t) &\left\{-\alpha \frac{\mu''(t)}{\mu(t)} + \alpha \left(\frac{\mu'(t)}{\mu(t)}\right)^2 - (\alpha+\beta)\delta D) \left(\frac{\mu'(t)}{\mu(t)}\right)^2\right\} \\ &- \alpha \frac{\mu'(t)}{\mu(t)} \Gamma'(t) + (\alpha+\beta)\delta D \left\{\frac{\mu''(t)}{\mu(t)} + (\beta-1)\left(\frac{\mu'(t)}{\mu(t)}\right)^2\right\} \\ &\times \int_t^\infty \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta} \frac{\mu'(\tau)}{\mu(\tau)} \Gamma(\tau)d\tau.\end{aligned} \tag{48}$$

By (47) and (48), we conclude that $\Gamma(t)$ satisfies the differential equation

$$\Gamma''(t) - f(t)\Gamma'(t) - g(t)\Gamma(t) = 0, \tag{49}$$

where

$$f(t) = \frac{\mu''(t)}{\mu'(t)} + (\beta-\alpha-1)\frac{\mu'(t)}{\mu(t)},$$

and

$$g(t) = \big(\alpha\beta - (\alpha+\beta)\delta D\big)\left(\frac{\mu'(t)}{\mu(t)}\right)^2.$$

We look for a solution of equation (49) in the form

$$\Gamma(t) = \mu(t)^{-\lambda}, \quad \lambda > 0. \tag{50}$$

Substituting (50) into (49), we find that λ satisfies the equation

$$\lambda^2 - (\alpha-\beta)\lambda + (\alpha+\beta)\delta D - \alpha\beta = 0. \tag{51}$$

Using condition (43), one can verify that equation (51) has the unique positive solution $\lambda = \tilde{\alpha}$. Therefore, equation (49) has the bounded solution $\mu(t)^{-\tilde{\alpha}}$, and in particular

$$\Gamma(t) = \Gamma(s)\left(\frac{\mu(t)}{\mu(s)}\right)^{-\tilde{\alpha}}.$$

Substituting $\Gamma(t)$ in (46) and setting $t = s$, we obtain

$$\begin{aligned}\Gamma(s) &\le D\gamma\nu(s)^\varepsilon + \delta D\Gamma(s)\int_s^\infty \left(\frac{\mu(\tau)}{\mu(s)}\right)^{-\beta-\tilde{\alpha}} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &= D\gamma\nu(s)^\varepsilon + \frac{\delta D}{\beta+\tilde{\alpha}}\Gamma(s),\end{aligned}$$

which implies that

$$\Gamma(s) = \frac{D}{1-\delta D/(\beta+\tilde{\alpha})}\gamma\nu(s)^\varepsilon = \tilde{D}_1\gamma\nu(s)^\varepsilon.$$

Therefore,

$$\Gamma(t) = \tilde{D}_1 \gamma \left(\frac{\mu(t)}{\mu(s)}\right)^{-\tilde{\alpha}} \nu(s)^{\varepsilon}.$$

One can now use a similar idea to that in the proof of Lemma 3 to show that $\varphi(t) \le \Gamma(t)$ for every $t \ge s$. $\square$

It follows from Lemma 6 that

$$\|\hat{P}(t)\hat{T}(t,s)\xi\| \le \tilde{D}_1 \left(\frac{\mu(t)}{\mu(s)}\right)^{-\tilde{\alpha}} \nu(s)^{\varepsilon} \|\hat{P}(s)\xi\|, \quad t \ge s.$$

Setting $\eta = \hat{P}(s)\xi$, we obtain

$$\|\hat{T}(t,s)\hat{P}(s)\eta\| = \|\hat{P}(t)\hat{T}(t,s)\xi\| \le \tilde{D}_1 \left(\frac{\mu(t)}{\mu(s)}\right)^{-\tilde{\alpha}} \nu(s)^{\varepsilon} \|\eta\|$$

for every $t \ge s$. This establishes inequality (41).

Now we show that inequality (42) holds. For each $t \le s$, we have

$$\begin{aligned} \hat{Q}(t)\hat{T}(t,s) = T(t,s)Q(s)\hat{Q}(s) &+ \int_0^t T(t,\tau)P(\tau)B(\tau)\hat{Q}(\tau)\hat{T}(\tau,s)\,d\tau \\ &- \int_t^s T(t,\tau)Q(\tau)B(\tau)\hat{Q}(\tau)\hat{T}(\tau,s)\,d\tau. \end{aligned} \tag{52}$$

Let $\xi \in X$. We set

$$\psi(t) = \|\hat{T}(t,s)\hat{Q}(s)\xi\| \quad \text{for} \quad t \le s,$$

and $\gamma = \|\hat{Q}(s)\xi\|$. By (52) and (39)–(40), the function ψ is bounded, and satisfies

$$\begin{aligned} \psi(t) &\le \|T(t,s)Q(s)\|\gamma + \int_0^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \psi(\tau)\,d\tau \\ &\quad + \int_t^s \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\|\psi(\tau)\,d\tau \\ &\le D\gamma \left(\frac{\mu(s)}{\mu(t)}\right)^{-\beta} \nu(s)^{\varepsilon} + \delta D \int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \frac{\mu'(\tau)}{\mu(\tau)}\psi(\tau)\,d\tau \\ &\quad + \delta D \int_t^s \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta} \frac{\mu'(\tau)}{\mu(\tau)}\psi(\tau)\,d\tau. \end{aligned}$$

Lemma 7. *We have*

$$\psi(t) \le \tilde{D}_2 \gamma \left(\frac{\mu(s)}{\mu(t)}\right)^{-\tilde{\alpha}} \nu(s)^{\varepsilon}, \qquad 0 < t \le s,$$

where

$$\tilde{D}_2 = \frac{D}{1 - \delta D/(\alpha + \tilde{\alpha})} > 0. \tag{53}$$

Proof of the lemma. Again we specify a bounded continuous function Ψ by requiring that it satisfies the integral equation

$$\begin{aligned}\Psi(t) = D\gamma \left(\frac{\mu(s)}{\mu(t)}\right)^{-\beta} \nu(s)^{\varepsilon} + \delta D \int_0^t \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha} \frac{\mu'(\tau)}{\mu(\tau)} \Psi(\tau) d\tau \\ + \delta D \int_t^s \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta} \frac{\mu'(\tau)}{\mu(\tau)} \Psi(\tau) d\tau\end{aligned} \tag{54}$$

for every $t \le s$. One can proceed in a similar manner to that in the proof of Lemma 6 to show that

$$\Psi(t) = \Psi(s) \left(\frac{\mu(s)}{\mu(t)}\right)^{-\tilde{\alpha}}.$$

Substituting $\Psi(t)$ in (54) and setting $t = s$, we obtain

$$\begin{aligned}\Psi(s) &= D\gamma\nu(s)^{\varepsilon} + \delta D \Psi(s) \int_0^s \left(\frac{\mu(s)}{\mu(\tau)}\right)^{-\alpha-\tilde{\alpha}} \frac{\mu'(\tau)}{\mu(\tau)} d\tau \\ &\le D\gamma\nu(s)^{\varepsilon} + \frac{\delta D}{\alpha + \tilde{\alpha}} \Psi(s).\end{aligned}$$

Hence, $\Psi(s) \le \tilde{D}_2 \gamma \nu(s)^{\varepsilon}$ and again one can proceed in a similar manner to that in the proof of Lemma 6 to obtain

$$\psi(t) \le \Psi(t) \le \tilde{D}_2 \gamma \left(\frac{\mu(s)}{\mu(t)}\right)^{-\tilde{\alpha}} \nu(s)^{\varepsilon}, \quad 0 < t \le s.$$

This completes the proof of the lemma. □

The lemma yields inequality (42), thus completing the proof of the theorem. □

9. Robustness of nonuniform dichotomies II

We remark that Theorem 5 gives no information about the norms of the projections $\hat{P}(t)$ and $\hat{Q}(t)$ for the perturbed equation. This is important in order to establish the robustness of nonuniform dichotomies. For that we require a slightly stronger hypothesis, namely

$$\|B(t)\| \le \delta \nu(t)^{-2\varepsilon} \left\{ (\beta + \tilde{\alpha}) \frac{\mu'(t)}{\mu(t)} + \varepsilon \frac{\nu'(t)}{\nu(t)} \right\}, \quad t \ge 0. \tag{55}$$

The following is our robustness result.

Theorem 6. *Let $A, B\colon \mathbb{R}_0^+ \to \mathcal{B}(X)$ be continuous functions such that equation* (1) *admits a (μ,ν)-nonuniform dichotomy with growth rates $\mu \geq \nu$ and $\varepsilon < \min\{\alpha,\beta\}$. If $B(t)$ satisfies* (55) *with δ sufficiently small, then equation* (2) *admits a (μ,ν)-nonuniform dichotomy with the constants α, β, D and ε replaced respectively by $\tilde{\alpha}, \tilde{\alpha}, 4D\tilde{D}$ and 2ε.*

Proof. Again we follow arguments in [6]. We start with an auxiliary result.

Lemma 8. *If $\delta < 1/4D\tilde{D}$, then for each $t \geq 0$ we have*

$$\|\hat{P}(t)\| \leq 4D\nu(t)^\varepsilon, \qquad \|\hat{Q}(t)\| \leq 4D\nu(t)^\varepsilon. \tag{56}$$

Proof of the lemma. By Lemma 5 with $t = s$, since $P(t)$ and $Q(t)$ are complementary projections, we have

$$Q(t)\hat{P}(t) = -\int_t^\infty T(t,\tau)Q(\tau)B(\tau)\hat{P}(\tau)\hat{T}(\tau,t)d\tau. \tag{57}$$

By (41) and (44), for each $\tau \geq t$ we have

$$\|\hat{P}(\tau)\hat{T}(\tau,t)\| \leq \tilde{D}\left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\tilde{\alpha}} \nu(t)^\varepsilon \|\hat{P}(t)\|. \tag{58}$$

Using (57) and (40), we obtain

$$\begin{aligned}
&\|Q(t)\hat{P}(t)\| \\
&\leq \int_t^\infty \|T(t,\tau)Q(\tau)\| \cdot \|B(\tau)\| \cdot \|\hat{P}(\tau)\hat{T}(\tau,t)\| d\tau \\
&\leq \delta D\tilde{D}\|\hat{P}(t)\| \int_t^\infty \left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta-\tilde{\alpha}} \left(\frac{\nu(t)}{\nu(\tau)}\right)^\varepsilon \left\{(\beta+\tilde{\alpha})\frac{\mu'(\tau)}{\mu(\tau)} + \varepsilon\frac{\nu'(\tau)}{\nu(\tau)}\right\} d\tau \\
&= -\delta D\tilde{D}\|\hat{P}(t)\| \int_t^\infty \left\{\left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\beta-\tilde{\alpha}} \left(\frac{\nu(t)}{\nu(\tau)}\right)^\varepsilon\right\}_\tau' d\tau \\
&= \delta D\tilde{D}\|\hat{P}(t)\|.
\end{aligned} \tag{59}$$

Similarly, it follows from (52) with $t = s$ that

$$P(t)\hat{Q}(t) = \int_0^t T(t,\tau)P(\tau)B(\tau)\hat{Q}(\tau)\hat{T}(\tau,t)d\tau. \tag{60}$$

By (42), for each $\tau \leq t$ we have

$$\|\hat{Q}(\tau)\hat{T}(\tau,t)\| \leq \tilde{D}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\tilde{\alpha}} \nu(t)^\varepsilon \|\hat{Q}(t)\|.$$

Now we observe that since $\mu \geq \nu$ and $\varepsilon < \alpha + \tilde{\alpha}$, we have

$$\mu(t)^{\alpha+\tilde{\alpha}} \geq \mu(t)^{\varepsilon} \geq \nu(t)^{\varepsilon}. \tag{61}$$

Using (60), (61) and (39), we obtain

$$\begin{aligned}
\|P(t)\hat{Q}(t)\| &\leq \int_0^t \|T(t,\tau)P(\tau)\| \cdot \|B(\tau)\| \cdot \|\hat{Q}(\tau)\hat{T}(\tau,t)\| d\tau \\
&= \delta D\tilde{D}\|\hat{Q}(t)\| \int_0^t \left\{ \left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\alpha-\tilde{\alpha}} \left(\frac{\nu(t)}{\nu(\tau)}\right)^{\varepsilon} \right\}'_{\tau} d\tau \\
&= \delta D\tilde{D}\|\hat{Q}(t)\| \left\{ 1 - \frac{\nu(t)^{\varepsilon}}{\mu(t)^{\alpha+\tilde{\alpha}}} \right\} \\
&\leq \delta D\tilde{D}\|\hat{Q}(t)\|.
\end{aligned} \tag{62}$$

We also observe that

$$\hat{P}(t) - P(t) = Q(t)\hat{P}(t) - P(t)\hat{Q}(t).$$

It thus follows from (60) and (62) that

$$\|\hat{P}(t) - P(t)\| \leq \delta D\tilde{D}(\|\hat{P}(t)\| + \|\hat{Q}(t)\|). \tag{63}$$

On the other hand, by (39)–(40) with $t = s$, we have

$$\|P(t)\| \leq D\nu(t)^{\varepsilon}, \qquad \|Q(t)\| \leq D\nu(t)^{\varepsilon}$$

It thus follows from (63) that

$$\begin{aligned}
\|\hat{P}(t)\| &\leq \|\hat{P}(t) - P(t)\| + \|P(t)\| \\
&\leq \delta D\tilde{D}(\|\hat{P}(t)\| + \|\hat{Q}(t)\|) + D\nu(t)^{\varepsilon}.
\end{aligned}$$

Since

$$\|\hat{Q}(t) - Q(t)\| = \|\hat{P}(t) - P(t)\|,$$

we also have

$$\|\hat{Q}(t)\| \leq \delta D\tilde{D}(\|\hat{P}(t)\| + \|\hat{Q}(t)\|) + D\nu(t)^{\varepsilon}.$$

Therefore,

$$\|\hat{P}(t)\| + \|\hat{Q}(t)\| \leq 2\delta D\tilde{D}(\|\hat{P}(t)\| + \|\hat{Q}(t)\|) + 2D\nu(t)^{\varepsilon},$$

and thus,

$$(1 - 2\delta D\tilde{D})\|\hat{P}(t)\| + \|\hat{Q}(t)\| \leq 2D\nu(t)^{\varepsilon}.$$

Since $\delta < 1/4D\tilde{D}$, we obtain

$$\|\hat{P}(t)\| + \|\hat{Q}(t)\| \leq 4D\nu(t)^{\varepsilon},$$

which yields the desired inequalities. $\square$

Using (58) and (56), for each $\tau \ge t$ we have

$$\begin{aligned}\|\hat{P}(\tau)\hat{T}(\tau,t)\| &\le \tilde{D}\left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\tilde{\alpha}} \nu(t)^{\varepsilon}\|\hat{P}(t)\| \\ &\le 4D\tilde{D}\left(\frac{\mu(\tau)}{\mu(t)}\right)^{-\tilde{\alpha}} \nu(t)^{2\varepsilon}.\end{aligned}$$

Similarly, using (59) and (56), for each $\tau \le t$ we have

$$\begin{aligned}\|\hat{Q}(\tau)\hat{T}(\tau,t)\| &\le \tilde{D}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\tilde{\alpha}} \nu(t)^{\varepsilon}\|\hat{Q}(t)\| \\ &\le 4D\tilde{D}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{-\tilde{\alpha}} \nu(t)^{2\varepsilon}.\end{aligned}$$

This completes the proof of the theorem. □

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