SAGBI bases for the Kernel of certain locally nilpotent $\mathbb{K}$-derivations on polynomial rings.

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Abstract. In this paper is obtained an algebraic-computational property related to the theory of SAGBI bases for the specific case of the kernel of some locally nilpotent derivations on the polynomial ring in several variables over a field of characteristic zero. The result achieved via explicit calculation is an algebraic characterization and a finite SAGBI basis for the kernel of a certain class of locally nilpotent derivations.

Introduction.

The symbol $\Bbbk$ will be used to indicate a field of characteristic zero, the symbols $x_1, \ldots, x_n$ will always indicate independent algebraic variables and will be adopted the notation $\underline{x}^a := x_1^{a_1} \ldots x_n^{a_n}$ for all $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Furthermore, the polynomial ring $\Bbbk[x_1, \ldots, x_n]$ will be denoted by $\Bbbk[\underline{x}]$ and its fraction field by $\mathbb{K}(\underline{x})$.

The ideals and subalgebras in $\mathbb{K}[\underline{x}]$ have been treated with more accurate and expressive methods since the heyday of computational algorithms in 70's. These algorithms were made to solve questions about these two objects and they are based on analogous structures: *Gröbner bases for ideals* and *SAGBI bases for subalgebras.* On a different way, a classical problem is to determine if the kernel of a derivation on a polynomial ring is a finitely generated $\mathbb{K}$-algebra. This problem is strongly connected with the *theory of invariant rings*

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and the historical *Fourteenth Hilbert's Problem.* The study of derivations on its current form came from the late 50's with many papers from great mathematicians like *Dixmier*, *Nouazé* and *Rentschler*, whose motivations came from Lie algebras and Lie groups, where there exist strong connections with derivations, especially locally nilpotent derivations. Recently many works are dedicated to algebraic-geometric-computational treatments for the kernel of derivations, invariant theory and SAGBI bases, for instance [1], [9] and [10].

This paper provides a link between computational properties of SAGBI bases and algebraic properties of $\mathbb{K}$-derivations on $\mathbb{K}[\underline{x}]$. The groundwork of all this process is a *Zariski's morphism associated to a derivation.*

1. **Elements of the SAGBI bases theory.**

This section has the aim to establish few notations and concepts which will be used along the paper. A recommended literature for greater details is the article [16], accompanied of [2] and [8].

A monomial order in $\mathbb{K}[\underline{x}]$ is a order coming from a total additive order in $\mathbb{Z}^n$ which turn $0 \in \mathbb{Z}^n$ the minimal element, i.e

$$\underline{x}^a \preceq \underline{x}^b \text{ in } \mathbb{K}[\underline{x}] \Longleftrightarrow a \preceq b \text{ in } \mathbb{Z}^n,$$

and a lexographic order in $\mathbb{K}[\underline{x}]$ is a monomial order such that $\underline{x}^a \preceq \underline{x}^b$ if, and only if, $b_\ell - a_\ell > 0$ for the maximal $\ell \in \{1, \ldots, n\}$ such that $b_\ell - a_\ell \neq 0$.

The set of all monomial orders in $\mathbb{K}[\underline{x}]$ will be denoted by Ω and its elements by $\preceq$. Given $f = \sum\limits_{a \in \mathcal{F} \subset \mathbb{Z}^n} c_a \underline{x}^a \in \mathbb{K}[\underline{x}]$, where $\mathcal{F}$ is a finite and non-empty set, the support of f is defined by $supp(f) = \{a \in \mathbb{Z}^n \mid c_a \neq 0\}$. Accordingly, if $f \neq 0$, then $supp(f)$ is a finite non-empty subset of $\mathbb{Z}^n$ and it makes possible to consider the maximum element of $supp(f)$. This element will be denoted by $max_\preceq(f)$. Further, we will denote the *leader term* of f by $lt_\preceq(f)$, i.e.

$$lt_\preceq(f) := \begin{cases} c_{max_\preceq(f)} \underline{x}^{max_\preceq(f)}, & \text{if} \quad f \neq 0 \\ 0, & \text{if} \quad f = 0. \end{cases}$$

Naturally, given a $\mathbb{K}$-vector subspace $V \subseteq \mathbb{K}[\underline{x}]$, will be denoted by $lt_\preceq(V)$ the $\mathbb{K}$-vector space generated by $\{lt_\preceq(f) \mid f \in V\}$. It is easy to see that

$$max_{\preceq}(fg) = max_{\preceq}(f) + max_{\preceq}(g), \quad \forall\, f, g \in \mathbb{K}\left[\underline{x}\right] \setminus \{0\}\,,$$

and

$$lt_{\preceq}(fg) = lt_{\preceq}(f) \cdot lt_{\preceq}(g), \quad \forall\, f, g \in \mathbb{K}\left[\underline{x}\right].$$

In the case that A is a $\mathbb{K}$-algebra, then $lt_{\preceq}(A)$ is also a $\mathbb{K}$-algebra.

Finally, the description of what is understood as a *SAGBI* basis for a $\mathbb{K}$-subalgebra of $\mathbb{K}\left[\underline{x}\right]$:

Definition 1.1. Fix an order $\preceq\, \in \Omega$. A subset S of a $\mathbb{K}$-subalgebra $A \subseteq \mathbb{K}\left[\underline{x}\right]$ is a SAGBI basis for A if S is a $\mathbb{K}$-generating set for A and satisfies

$$lt_{\preceq}(A) = \mathbb{K}\left[\{lt_{\preceq}(s) \mid s \in S\}\right].$$

Furthermore, S is said to be a finite SAGBI basis for A if S is a finite set.

Remark. A SAGBI basis depends extremely of the fixed order. For example, different orders can result in bases with different cardinality. Moreover, there exist subalgebras with no basis and subalgebras with only infinite bases for any choice of order.

2. **Derivations over rings.**

In this section we review the concept of derivations and locally nilpotent derivations over a ring with fundamental properties. The main references of this part are [3], [6], [13] and [14].

Definition 2.1. A derivation D over a ring A is an application $D : A \to A$ satisfying:

(1) $D(x+y) = D(x) + D(y)$ and
(2) $D(xy) = D(x)y + xD(y)$, $\forall\, x, y \in A$.

When A is a $\mathbb{K}$-algebra, a derivation D over A is called a $\mathbb{K}$-derivation if

$$D(\alpha x) = \alpha D(x), \forall\, \alpha \in \mathbb{K} \text{ and } x \in A.$$

The kernel of a derivation D over a ring A is the subset of A defined by

$$\ker(D) := \{x \in A \mid D(x) = 0\}\,.$$

Note that if A is a $\mathbb{K}$-algebra and D is a $\mathbb{K}$-derivation, then $\mathbb{K} \subseteq \ker(D)$ and $\ker(D)$ is a $\mathbb{K}$-subalgebra of A. The fundamental example of a derivation is the following.

Example 2.1. The partials derivatives $\frac{\partial}{\partial x_i}$ defined in $\mathbb{K}[\underline{x}]$ are derivations in $\mathbb{K}[\underline{x}]$. Besides, if $f \in \mathbb{K}[\underline{x}]$, then $D = f\frac{\partial}{\partial x_i}$ is also a derivation in $\mathbb{K}[\underline{x}]$.

Some of the most important properties of the theory, whose proofs are extremely direct, are:

(1) The set $der(A)$ of derivations of a ring A is an A-module.
(2) The set $der_{\mathbb{K}}(A)$ of $\mathbb{K}$-derivations over a $\mathbb{K}$-algebra A is a $\mathbb{K}$-submo-
dule of $der(A)$.
(3) The module $der_{\mathbb{K}}(\mathbb{K}[\underline{x}])$ is a free $\mathbb{K}[\underline{x}]$-module with basis given by $\frac{\partial}{\partial x_i}$ for $i = 1, ..., n$.

The following example illustrates a rich property about the behavior of iterations of some derivations.

Example 2.2. Set $D := \frac{d}{dt} : \mathbb{R}[t] \to \mathbb{R}[t]$. A very simple fact is that for each $f \in \mathbb{R}[t]$ such that $\deg(f) = n$, we have $D^{n+1}(f) = 0$. However, when we consider the $\mathbb{K}$-derivation $tD = t\frac{d}{dt} : \mathbb{R}[t] \to \mathbb{R}[t]$ it does not happen for all elements, for instance $D^n(t) = t \neq 0$, for all $n \in \mathbb{N}$.

Next follows the definition which formalize this phenomenon:

Definition 2.2. A derivation $D : A \to A$ is called locally nilpotent if, for each $x \in A$, there exists $n \in \mathbb{N}$ such that $D^n(x) = 0$.

The set of all locally nilpotent derivations over A will be denoted by $lnd(A)$ and the set of all $\mathbb{K}$-locally nilpotent derivations will be denoted by $lnd_{\mathbb{K}}(A)$. A well known and intuitive fact is that the partial derivatives $\frac{\partial}{\partial x_i} : \mathbb{K}[\underline{x}] \to \mathbb{K}[\underline{x}], i = 1, \ldots, n$, introduced in the example 2.1, are locally nilpotent $\mathbb{K}$-derivations.

An important tool associated with a derivation is called *exponential map*. To complete this section, this application will be built and also will be established some of the immediate consequences which will be used posteriorly.

2.1. **The exponential map.** Let A be a $\mathbb{K}$-algebra and $D \in der(A)$. Consider the formal power series ring $A[[t]]$ in only one variable t over

A and extend the derivation D to a derivation $\widetilde{D}$ over $A[[t]]$ by the formula

$$\widetilde{D}\left(\sum a_i t^i\right) := \sum D(a_i)t^i.$$

Now, set the map

$$e^{tD} : A[[t]] \to A[[t]]$$

by

$$e^{tD}(f) := \sum_{p=0}^{\infty} \frac{1}{p!}\widetilde{D}^p(f)t^p, \ \forall\, f \in A[[t]].$$

Definition 2.3. The map e^{tD} is called exponential map associated to the derivation D.

The importance of the function e^{tD} comes from the following well known result.

Proposition 2.4. The map $e^{tD} : A[[t]] \to A[[t]]$ is an automorphism of the ring $A[[t]]$. In particular, if D is a locally nilpotent derivation of A, then the map $e^{tD} : A[t] \to A[t]$ is an automorphism of the polynomial ring $A[t]$.

2.2. **Zariski map and its consequences.** In many branches of the mathematics is important to analyze the structure of the particular element defined by a $\mathbb{K}$-derivation: its kernel. In this section we will focus on properties about the kernel of a locally nilpotent $\mathbb{K}$-derivation.

Suppose that A is a domain of characteristic zero. If D is a locally nilpotent derivation of A and $b \in A$, then can be defined an endomorphism of A by formula

$$\psi_b(a) := \sum_{i=0}^{n} \frac{1}{i!}D^i(a)b^i, \quad \forall\, a \in A,$$

where $n = \max\{n \in \mathbb{N} \mid D^n(a) \neq 0\}$. This homomorphism is called *Zariski map associated to D and b.*

From this point there exist few results about locally nilpotent derivations to characterize the elements of the ring on which they are being considered, in particular for their kernels. The next results are well-known, for more details see [6] and [17].

Lemma 2.5. If A is a domain of characteristic zero, $D \in lnd(A)$ and $s \in A$ (arbitrary), then every element $a \in A$ is a polynomial in s of the form $a = \sum_{i=0}^{\infty} \frac{1}{i!}\psi_{-s}(D^i(a))s^i$.

Proof. Let $a \in A$. Then, $e^{tD}(a) \in A[t]$ and let us assume that its degree in t is N. It follows that

$$\begin{array}{rcl} a & = & e^{-tD} \circ e^{tD}(a) \\ & = & \sum_{i=0}^{N} \frac{1}{i!} e^{-tD}(D^i(a))t^i, \end{array}$$

where we used the proposition 2.4 and the fact that $e^{-tD}(t^i) = t^i$, for all $i \in \mathbb{N}$. Finally, replacing t by s, we have $a = \sum_{i=0}^{\infty} \frac{1}{i!}\psi_{-s}(D^i(a))s^i$. □

The next result is often attributed to *Wright* in [19] (section 2.1), however this result already appeared in a paper by *Gabriel* and *Nouazé* in [7] about 14 years earlier. Its proof depends of the previous lemma.

Theorem 2.6. If A is a $\mathbb{Q}$-algebra, $D \in lnd(A)$ and there exists $s \in A$ such that $D(s) = 1$, then A is a polynomial ring in s over $\ker(D)$, i.e. $A = \ker(D)[s]$.

Corollary 2.7. If A is a R-algebra, R is a $\mathbb{Q}$-algebra, $D \in lnd_R(A)$ and there exists $s \in A$ such that $D(s) = 1$, then $\ker(D) = \psi_{-s}(A)$. In particular, if G is a generator set for the R-algebra A, then $\psi_{-s}(G)$ is a generator set for the R-algebra $\ker(D)$.

Proof. By theorem 2.6, we have $\psi_{-s}(A) \subset \ker(D)$. Conversely, if $a \in \ker(D)$, then $a = e^{tD}(a)$, whence $a = \psi_{-s}(a) \in \psi_{-s}(A)$. Hence, $\ker(D) = \psi_{-s}(A)$. □

Corollary 2.8. If D is a locally nilpotent non-trivial $\mathbb{K}$-derivation on $\mathbb{K}[\underline{x}]$. Then, there exists $s \in \mathbb{K}(\underline{x})$ such that

$$\ker(D) = \mathbb{K}[\psi_{-s}(x_1), \ldots, \psi_{-s}(x_n)] \cap \mathbb{K}[\underline{x}].$$

Proof. Let $f \in \mathbb{K}[\underline{x}] \setminus \ker(D)$, then there exists $l \in \mathbb{N}, l > 1$, such that $D^l(f) \neq 0$ and $D^{l+1}(f) = 0$. Consequently, if $s = \frac{D^{l-1}(f)}{D^l(f)}$, it follows that

$$D(s) = D\left(\frac{D^{l-1}(f)}{D^l(f)}\right) = \frac{D^l(f)D^l(f) - D^{l+1}(f)D^{l-1}(f)}{D^l(f)D^l(f)} = 1.$$

By corollary 2.7 we are done. □

The next example illustrates this corollary and it also shows that its application is algorithmic.

Example 2.3. Consider the following $\mathbb{K}$-derivation on $\mathbb{K}[x,y,z]$:

$$D = \frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y} + (x+y+xy^2)\frac{\partial}{\partial z}.$$

This derivation satisfies $D(x) = 1 \in \mathbb{K}$, $D(y) = x^2 \in \mathbb{K}[x]$ and $D(z) = x + y + xy^2 \in \mathbb{K}[x,y]$, it means that D is a locally nilpotent $\mathbb{K}$-derivation and, hence, the last corollary can be used and it will be made next.

Choose an explicit element $f \in \mathbb{K}[x,y,z] \setminus \ker(D)$ such that $D^{i+1}(f) = 0$ for some $i > 0$. For instance, $f = y$. Note that $D^4(y) = 0$. Following the previous notation, let $s \in \mathbb{K}[x,y,z]$ defined by $s := \frac{D^2(y)}{D^3(y)} = x$.

Therefore, a Zariski map can be write as

$$\psi_{-s}(g) = \psi_{-x}(g) = \sum_{i=0}^{\infty} \frac{(-x)^i}{i!}D^i(g), \quad \forall\, g \in \mathbb{K}[\underline{x}].$$

So, calculating $\psi_{-x}(x)$, $\psi_{-x}(y)$ and $\psi_{-x}(z)$, we have

$$\psi_{-x}(x) = 0, \qquad \psi_{-x}(y) = \tfrac{3y-x^3}{3},$$

$$\psi_{-x}(z) = -20x^2 + 10x^4 - x^8 - 40xy - 20x^2y^2 + 8x^5y + 40z.$$

Finally, we obtain

$$\ker(D) = \mathbb{K}\left[-20x^2 + 10x^4 - x^8 - 40xy - 20x^2y^2 + 8x^5y + 40z, 3y - x^3\right].$$

3. **The kernel of certain locally nilpotent $\mathbb{K}$-derivations.**

The goal of this section is to establish a simple presentation for the kernel of a specific type of locally nilpotent $\mathbb{K}$-derivation on $\mathbb{K}[\underline{x}]$. Therefore, will be achieved a finite SAGBI basis for the kernel of these derivations. All constructions depend of the map ψ_b defined on the section 2.2 and of a "good choice" of the multiplicative order in $\mathbb{K}[\underline{x}]$.

Theorem 3.1. Let D be a $\mathbb{K}$-derivation on $\mathbb{K}[\underline{x}]$ of the form

$$D = k\frac{\partial}{\partial x_1} + p_2\frac{\partial}{\partial x_2} + \cdots + p_n\frac{\partial}{\partial x_n},$$

where $p_i \in \mathbb{K}[x_1, \ldots, x_{i-1}]$, $2 \le i \le n$, and $k \in \mathbb{K}\setminus\{0\}$. Then,

$$\left\{\psi_{-\frac{x_1}{k}}(x_2), \ldots, \psi_{-\frac{x_1}{k}}(x_n)\right\}$$

is an explicit SAGBI basis for $\ker(D)$ regarding the lexicographic order $\prec \in \Omega$ such that

$$x_1 \prec \cdots \prec x_n.$$

Proof. It is clear that this sort of $\mathbb{K}$-derivation is locally nilpotent (it is a triangular derivation). Observe that for $i \in \{2, \ldots, n\}$ we have

$$\psi_{-\frac{x_1}{k}}(x_i) = \sum_{j=0}^{\infty} \frac{\left(-\frac{x_1}{k}\right)^j}{j!} D^j(x_i) = x_i + \sum_{j=1}^{\infty} \frac{\left(-\frac{x_1}{k}\right)^j}{j!} D^j(x_i).$$

However, by the choice of the p_i we conclude that $D^j(x_i)$ can involve only $x_1, \ldots, x_{i-1}$, for any $j \ge 1$ ($j \in \mathbb{N}$), and $i \in \{2, \ldots, n\}$. Hence,

$$lt_{\prec}\left(\psi_{-\frac{x_1}{k}}(x_i)\right) = x_i, \quad \text{for all } i \in \{2, \ldots, n\},$$

since $x_1 \prec \cdots \prec x_{i-1} \prec x_i$.

By corollary 2.8, $S = \{\psi_{-s}(x_2), \ldots, \psi_{-s}(x_n)\}$ generates $\ker(D)$ over $\mathbb{K}$, because for $s = \frac{x_1}{k} \in \mathbb{K}[\underline{x}]$ we have $D(s) = 1$, $\psi_{-s}(x_1) = 0$ and $\psi_{-s}(x_i) \in \mathbb{K}[\underline{x}]$ for all $i \ne 1$. Thus, $\mathbb{K}[x_2, \ldots, x_n] \subseteq lt_{\prec}(\ker(D))$. Therefore, S is a SAGBI basis for $\ker(D)$ if, and only if, we have

$$\mathbb{K}[lt_{\prec}(S)] = lt_{\prec}(\ker(D)).$$

Observe that

$$\mathbb{K}[lt_{\prec}(S)] = \mathbb{K}[lt_{\prec}(\psi_{-s}(x_2)), \ldots, lt_{\prec}(\psi_{-s}(x_n))] = \mathbb{K}[x_2, \ldots, x_n].$$

Hence, for S to be a SAGBI basis for $\ker(D)$ it is sufficient to show that

$$lt_{\prec}(\ker(D)) \subseteq \mathbb{K}[x_2, \ldots, x_n].$$

In order to do this we will prove that if $f \in ker(D)$, then $lt_{\prec}(f)$ does not involve x_1 and we are over.

Let $f \in Ker(D)$ and write

$$f = cx_1^{a_1} \dots x_n^{a_n} + p(x_1, ..., x_n),$$

for some $p \in \mathbb{K}[\underline{x}]$ such that $lt_{\prec}(f) = cx_1^{a_1} \dots x_n^{a_n}$, where $c \in \mathbb{K}$, $c \neq 0$. Suppose that $a_1 \neq 0$. Then, applying D on f we get

$$\begin{aligned} D(f) &= 0 \\ &= kca_1 x_1^{a_1-1} x_2^{a_2} \dots x_n^{a_n} + \\ &\quad + \textstyle\sum_{i=2}^{n} ca_i p_i x_1^{a_1} \dots x_{i-1}^{a_{i-1}} x_i^{a_i-1} x_{i+1}^{a_{i+1}} \dots x_n^{a_n} + D(p). \end{aligned} \tag{3.1}$$

The degree of $kca_1x_1^{a_1-1}x_2^{a_2}\dots x_n^{a_n}$ in the variable x_1 is

$$\deg_{x_1}(ca_1x_1^{a_1-1}x_2^{a_2}\dots x_n^{a_n}) = a_1 - 1$$

and each non-zero summand of

$$\sum_{i=2}^{n} ca_i p_i x_1^{a_1} \dots x_{i-1}^{a_{i-1}} x_i^{a_i-1} x_{i+1}^{a_{i+1}} \dots x_n^{a_n}$$

has degree in x_1 equal to

$$a_1 + \deg_{x_1}(p_i)$$

and in this case it is always bigger than $a_1 - 1$. So, there is no term in this summand to cancel with $kca_1x_1^{a_1-1}x_2^{a_2}\dots x_n^{a_n}$. Following the same idea, let $c_b x_1^{b_1}\dots x_n^{b_n}$ be an arbitrary monomial of p, then

$$(b_1, \dots, b_n) \prec (a_1, \dots, a_n) \tag{3.2}$$

and

$$\begin{aligned} D(c_b x_1^{b_1} \dots x_n^{b_n}) &= kc_b b_1 x_1^{b_1-1} x_2^{b_2} \dots x_n^{b_n} + \\ &\quad + c_b \textstyle\sum_{i=2}^{n} b_i p_i x_1^{b_1} \dots x_{i-1}^{b_{i-1}} x_i^{b_i-1} x_{i+1}^{b_{i+1}} \dots x_n^{b_n}. \end{aligned}$$

Again, we have the following two situations:

If $b_1 - 1 = a_1 - 1$, then $a_1 = b_1$ and the relation (3.2) implies that $a_k > b_k$ for some $k \in \{2, \dots, n\}$ and in this case this term cannot be canceled with $kca_1x_1^{a_1-1}x_2^{a_2}\dots x_n^{a_n}$.

If for some i the term $w := c_b b_i p_i x_1^{b_1} \dots x_{i-1}^{b_{i-1}} x_i^{b_i-1} x_{i+1}^{b_{i+1}} \dots x_n^{b_n}$ with $b_i \neq 0$ satisfies $\deg_{x_1}(p_i) + b_1 = a_1 - 1$, then w has degree in the variable x_i equals to $b_i - 1$. However, if in (3.2) we had either $a_i = b_i$ or $a_i > b_i$ for this i, then for both cases $a_i > b_i - 1$ and it is clear that w cannot be canceled with $kca_1x_1^{a_1-1}x_2^{a_2}\dots x_n^{a_n}$. On other hand, if it were $a_i < b_i$, we would have $a_k > b_k$ for some $k \in \{i+1, \dots, n\}$, so that w also could not be canceled with $kca_1x_1^{a_1-1}x_2^{a_2}\dots x_n^{a_n}$. Thus we have a contradiction with (3.1) coming from the hypothesis that $a_1 \neq 0$. Hence, $a_1 = 0$. Therefore, if $f \in ker(D)$, then $lt_{\prec}(f)$ does not involve the variable x_1. $\square$

This theorem shows via an explicit straightforward calculation that we have a finite SAGBI basis constructed from the Zariski map together of a natural multiplicative order on $\mathbb{K}[\underline{x}]$. It offers an algebraic-computational simplification to find the kernel of a derivation with some hypothesis that are not difficult to verify, neither unusual.

Returning to the example 2.3, note that it satisfies the conditions of the previews theorem. Thereby, the set of generators obtained in that example is a SAGBI basis regarding the lexicographical order $\prec$ such that $x \prec y \prec z$.

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