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On visual representations of Lie algebras: the type A_l between the old and the new

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Abstract. This paper is about some graph isomorphisms between the Auslander-Reiten quivers of the path algebras of quivers with underlying Dynkin diagrams of type A_l, and the weight diagrams or the weight graphs relative to some basic (adjoint) representations of a semi-simple complex Lie algebra associated to the same Dynkin diagrams. The crucial point is that, in the attempt to establish the above relationships, we constructed the oriented weight diagrams and the oriented weight graphs with respect to particular orientations on the Dynkin diagrams of type A_l. Our main goal is to furnish another application of weight theory and visual representations.

Introduction

Let K be an algebraically closed field of characteristic zero, say it $\mathbb{C}$. Let's consider the Auslander-Reiten quivers with respect to the quivers with underlying Dynkin diagrams of type A_l, $l \geq 1$. The aim of this essay is to produce graph isomorphisms between the Auslander-Reiten quivers above and the weight diagrams or the weight graphs associated to some basic (or adjoint) representations of a semi-simple complex Lie algebra of type A_l, $l \geq 1$.

The reader can find in Sections 1.1 and 1.2 a recall of some facts about roots, weights and representations of a semi-simple complex Lie algebra. The crystal graphs, that is the weight diagrams, of basic or adjoint (not basic) representations of complex simple Lie algebras, are those pictures for the visualization of calculations involving Lie algebras, Chevalley groups and the world around them. They are diagrams with nodes, which stand

Key words: Semi-simple Complex Lie Algebras; Basic Representations; Adjoint Representations; Crystal Graphs; Weight Graphs; Quivers; Auslander-Reiten Quivers.

for the weights of a representation and labeled edges which join two nodes/ weights, provided them differ by a fundamental root. At first C. W. Curtis, N. Iwahori and R. Kilmayer [5], used in their paper some of these pictures to find the decomposition of the Weyl group into double cosets. From then on these graphs have been pictured in various contexts. For a historical and methodological description of the argument, for the visualization of many pictures, for the application of the weight diagrams and for many additional references the reader is referred to some papers by N. A. Vavilov [27]-[30], by E. Plotkin, A. Semenov and N. A. Vavilov [23] and by C. Parker and G. Rohrle [22]. Our weight diagrams are the colored graphs in the terminology of M. Kashiwara. The crystal graphs with orientations were found by M. Kashiwara [15]- [17], and by he himself and N. Nakashima [18]. They are linked with the path model of P. Littelmann and the canonical bases of G. Lusztig, as one can find in [20, 10].

It is notorious that to formulate and to solve certain vector spaces problems, one can adopt some facts about Weyl groups and Dynkin diagrams. This idea was first studied by P. Gabriel in early 1970-ies. In attempt to simplify the procedure, he introduced both the notion of quiver and of its representation. Up till now, these concepts are one of the fundamental instruments in the representation theory. Briefly, a quiver is given by a pair of two sets: the set of vertices and the set of arrows with starting points and ending points in the set of vertices. Thus any quiver is an oriented graph. An Auslander-Reiten quiver is a picture that is a visualization for calculations of all indecomposable modules of an algebra of finite representation type (that is, with only a finite number of non isomorphic indecomposable modules). For our purpose, we consider only quiver of type A_l, without any loops or relations. They are schematically described in Examples 1.1-1.3.
Return to a general contest. In the literature there are interesting results about the connection between the representations of quivers and some structures of the corresponding Lie or Kac-Moody algebras. We apologize for the obvious uncompleteness of our citations. The starting point is, historically, P. Gabriel [7], who showed that
$(i)_G$: a quiver Γ without relations is of finite representation type if and only if its underlying graph is a disjoint union of Dynkin diagrams of type A_l, D_l, E_6, E_7, E_8;
$(ii)_G$: if Γ is as in $(i)_G$ then there exists a natural one to one correspondence, given by $\underline{\dim}$, between the indecomposable representations of Γ, and the positive roots related to the Dynkin diagram of an algebra of type A_l, D_l, E_6, E_7, E_8.
(see Section 1.3 for the definition of $\underline{\dim}$).
I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev [2], comparing the representations of quivers and the set of positive roots of a Lie algebra, produced another proof of $(ii)_G$, by means of the Coxeter functors. Moreover,

in the same paper they made some conjectures. We restate the conjecture (2) as follows:
(C): If α is a positive root then there exists exactly one indecomposable object V of Γ of type of dimension $\underline{\dim} V = \alpha$, to within isomorphisms.
K. Reineke [24], used representations of quivers to realize the crystal graphs of arbitrary irreducible representations of semi simple Lie algebras and so proved (2). V. G. Kac used the properties of infinite root systems in the representations of super-algebras; in particular in [11], the author gave another proof of $(ii)_G$ for the quivers of infinitely many indecomposable representations for a quiver without loops via root systems. Next some proofs were simplified in [12], where the root systems are interpreted via the representations of quivers. At last in [13], Kac himself compared the representations of quivers and the root structure of Kac-Moody algebras. The reader is referred also to [14], to contextualize better the infinite dimensional Lie algebras. This is the third edition of the original one in 1983, in which the author described deeply the Kac-Moody algebras; here one can find new developments, as the connections to mathematical physics. A. Odesskii and V. Sokolov [21], described some M-structures related to affine Dynkin diagrams of type A, D, E and their representations by affine quivers representations, see also [6, 26, 25].

The paper is organized as follows. The first Section is about the background. This is also to fix some of the notations, that we'll use, while some of the others standard and the new owns ones are given from time to time, in proper places. In Section two, we give a different proof both of $(ii)_G$ and of (C) for $l \geq 1$ (Cf. Lemma 2.1 and Corollary 2.1).
In Section three we observe that if a quiver Γ has a linear orientation, then the Auslander-Reiten quiver of its path algebra looks similar to the crystal graph of second exterior power of the natural representation of the semisimple complex Lie algebra A_l. Thus we construct an orientation on the crystal graph and find an isomorphism for $l \geq 2$ (Cf. Construction 3.2 and Theorem 3.1). The Section four is devoted to the case where the quiver Γ has a non linear orientation, We construct a new weight graph and find the graph isomorphism for $l \geq 3$ (Cf. Constructions 4.1, 4.2 and Theorem 4.1).

Our proofs are based on simple exercises of linear algebra. But we retain that another approach is possible. So in Section 1.4, we report the fundamental definitions about categories and functors, and in Remarks 2.1, 3.1, 4.1, we sketch this other procedure. The details are left to the interest reader.

<u>A remark</u>. To the author's knowledge, only $(ii)_G$ has been proved, as it was explained some few lines above. Moreover, we believe that there are

not explicit constructions of oriented weight diagrams and weight graphs as the our ones (Sections 3 and 4). At the same time, we are sure that much of the concepts included in this paper are known to a great many readers, but our main interest is the utmost simplicity of the explicit proofs for the interpretation of the Auslander-Reiten quivers via some representations of a semi simple Lie algebra, at least for the type A_l. In the light of what we have just written, we retain that our work is both original and a largely expository paper too. We think that this essay may be an amusing reading, especially for not experted readers, to whom is devoted also our long Section 1.

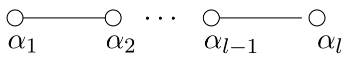
1. **Background and notations**

1.1. **Roots, Weyl groups and weights.** In this Section we report most of the notations, that we'll use.

We suppose that the reader is familiar with some concepts as those of semi-simple complex Lie algebra L and its Cartan subalgebra H. However we recall some general facts on the abstract root system and the abstract weights, for more details we remand to the standard literature [3, 4, 9].
Let $\emptyset \neq \Phi \subseteq V$ be a reduced, irreducible root system of L with respect to H, $rank(\Phi) = l$. We denote by $\check{\Phi} = \{\check{\alpha} = 2\alpha|(\alpha,\alpha) : \alpha \in \Phi\}$ the set of the co-roots. If we fix a lex order on Φ, once and for all, then Φ^+, Φ^- and $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ denote, respectively, the sets of positive, negative and fundamental roots. For any $\alpha \in \Phi$, one has $\alpha = \sum_{i=1}^{l} \lambda_i \alpha_i$, such that either $\lambda_i \geq 0$ for all i or $\lambda_i \leq 0$ for all i, $\alpha_i \in \Pi$. In particular if $\lambda_i = 1$, for all i, $\tilde{\alpha} = \sum_{i=1}^{l} \alpha_i$, for $\alpha_i \in \Pi$ is called the maximal root of Φ. If $\lambda_i \geq 0$ for all i, we define $ht(\alpha)$ the height of α by $ht(\alpha) = \sum_{i=1}^{l} \lambda_i$. One has $ht(\alpha) = 1$ for all $\alpha \in \Pi$, $ht(\tilde{\alpha}) = l$, and for each non fundamental positive root there exists another positive root of smaller height.
Let $W = W(\Phi) =< w_\alpha, \alpha \in \Pi >$ denote the group of the orthogonal reflection with respect to $\alpha_i \in \Pi$, it is called the Weyl group of Φ. (If X is a set, $< X >$ denotes the subgroup generated by X).
The Dynkin diagram of Φ is a graph with edges and nodes; the edges stand for the angle between two subsequent fundamental roots that are the nodes of the diagram. We'll indicate the Dynkin diagram of Φ by D_Φ.
$Q(\Phi) = \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus ... \oplus \mathbb{Z}_{\alpha_l}$ is called the root lattice;
$Q(\check{\Phi}) = \mathbb{Z}_{\check{\alpha}_1} \oplus \mathbb{Z}_{\check{\alpha}_2} \oplus ... \oplus \mathbb{Z}_{\check{\alpha}_l}$ is called the co-root lattice;
$P(\Phi) = (Q(\check{\Phi}))^* = \{\lambda \in V| < \lambda, \check{\beta} > \in \mathbb{Z}, \forall \beta \in \Phi\}$ is called the weight lattice. (If L is a lattice of V, $L^* = \{v \in V|(u,v) \in \mathbb{Z}, \forall u \in L\}$).
An element $\lambda \in P(\Phi)$ is called an integral weight of Φ. We remark that for if $< \lambda, \check{\alpha} >= \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, then any root is an integral weight.

Since the set of fundamental weights is given by $\check{\Pi} = \{\bar{\omega}_i | (\bar{\omega}_i, \check{\alpha}_j) = \delta_{ij}\}$, then one has $P(\Phi) = \mathbb{Z}_{\bar{\omega}_1} \oplus \mathbb{Z}_{\bar{\omega}_2} \oplus ... \oplus \mathbb{Z}_{\bar{\omega}_l}$.
$P(\Phi)_{++} = \{\mu \in P(\Phi) | (\mu, \alpha) > 0, \alpha \in \Pi\} = \{\sum m_i \bar{\omega}_i | m_i \in \mathbb{Z}, m_i \geq 0\}$ is called the set of the dominant integral weights. The choice of the fundamental system Π defines a partial order on $P(\Phi)$ such that $\lambda_1 \geq \lambda_2$ if and only if $\lambda_1 - \lambda_2 = \sum_{i=1}^{l} \alpha_i$, $\alpha_i \in \Pi$.
In the case of our interest, that is $\Phi = A_l$, $L \cong \mathfrak{sl}(l+1, \mathbb{C})$, $l \geq 1$ we follow the constructions in [3].
Let us consider $\mathbb{R}^{l+1}$ with an orthonormal basis $\{e_1, ..., e_{l+1}\}$ and V be the hyperplane in the space orthogonal to the vector $e_1 + ... + e_{l+1}$.
We define $\Phi = \{\alpha \in V | (\alpha, \alpha) = 2\}$, thus the roots have all the same length. We have $\Phi = \{e_j - e_j, 1 \leq i \neq j \leq l+1\}$, we can choose as fundamental system $\Pi = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, ..., \alpha_l = e_l - e_{l+1}\}$, and the maximal root is given by $\tilde{\alpha} = \alpha_1 + \alpha_2 + ... + \alpha_l = e_1 - e_{l+1}$,

D_{A_l} :



with the angle between α_i and α_{i+1}, $1 \leq i \leq l-1$, equal to $\theta = 2\pi/3$.
If $\alpha = e_i - e_j$, the action of w_α on the elements of the basis is given by :

$$w_\alpha(e_h) = \begin{cases} e_i & \text{if } h = j \\ e_j & \text{if } h = i \\ e_h & \text{if } h \neq i, j \end{cases}, \quad 1 \leq i \neq j \neq h \leq l+1$$

The fundamental weights are: $\bar{\omega}_i = e_1 + e_2 + ... + e_i - \frac{i}{l+1}(e_1 + e_2 + ... + e_{l+1})$, $1 \leq i \leq l$.
In direct calculations, we omit the projection on the hyperplane, so that: $\bar{\omega}_i = e_1 + e_2 + ... + e_i$, $1 \leq i \leq l$.

1.2. **Representations, crystal graphs and weight graphs.** For the part relative to representations we report heavily some of the results in [23]. Let V be a vector space over $\mathbb{C}$, L be a semi-simple complex Lie algebra and $\pi : L \longmapsto gl(V)$ be a representation of L over $\mathbb{C}$ in V. For a $\lambda \in H^*$, $V^\lambda = \{v \in V \ \pi(h)v = \lambda(h)v, h \in H\}$ is a subgroup of the space V, regarded as H-module, and it is called the space of weight λ. If $V^\lambda \neq \{0\}$ then λ is a weight of π with multiplicity $m_\lambda = \dim V^\lambda$.
Let $\bar{\Lambda}(\pi)$ be the set of weights of π, let $\Lambda(\pi)$ be the set of weights of π with multiplicity, let $(\bar{\Lambda}(\pi))^*$ be the set of non zero-weights of π, let $(\Lambda(\pi))^*$ be the set of non zero-weights of π with multiplicity. $P(\pi)$ denotes the lattice of the representation π and one has $V = \oplus V^\lambda$, $\lambda \in \Lambda(\pi)$.
Given a $\lambda \in \Lambda(\pi)$ and $v^+ \in V$, λ is called the highest weight of π, of multiplicity 1 and v^+ is called the primitive vector, viz. the highest weight vector, if $\pi(e_\alpha)v^+ = 0$, $\alpha \in \Phi^+$.
If V is irreducible, there is just one primitive element v^+ up to scalar.

The representation π is called: irreducible if and only if V is generated as L-module by a vector of highest weight;
basic if it is irreducible and the Weyl group acts transitively on $(\Lambda(\pi))^*$, i. e. for $\lambda, \mu \in (\Lambda(\pi))^*$ such that $\lambda - \mu = \alpha \in \Pi$ then $w_\alpha \lambda = \mu$, $w_\alpha \in W(\Phi)$;
adjoint ($\pi = ad$) if $V = L$, $(\Lambda(\pi))^* = \Phi$, $\Lambda(\pi) = \Phi \cup \{0_1, ..., 0_l\}$, $P(\pi) = Q(\Phi)$, $V^\alpha = L_\alpha$, for $\alpha \in \Phi$, $V^0 = H$, where L_α are the root subspace for $\alpha \in \Phi$[1].
There is an isomorphism between the set of isomorphism class of irreducible L-modules of finite dimension and the set of dominant integral weights.
We consider the adjoint and basic representations: in these cases all weights different from zero have multiplicity 1 and the use of the crystal graphs is powerful.
The Crystal graphs or weight diagrams are diagrams with nodes and edges, we usually read them from right to left and from bottom to top, since a larger weight stands to the left and to a higher position than a smaller one. To each weight different from zero there is a unique node. An edge joins the weights λ, μ if $\lambda - \mu = \alpha_i \in \Pi$ and it is labeled by i or by α_i. Moreover the diagrams are such that the labels on the opposite edges are equal and we write them once only. We use a version with no-oriented labels [23].
The weight graphs are graphs that have often very strong symmetries, with nodes and edges. We can associate them with every representation of L, but in this case two weights (nodes) are linked by an edge if their difference is a root.
The crystal graph of the representation $(\Phi, \bar{\omega}_i)$ is drawing starting from the fundamental weight $\bar{\omega}_i$.
Our interest is for $\Phi = A_l$, for $l \geq 1$. Look at Figure 1, where the left-hand side is the representation and the right-hand side is its weight diagram. We have just written how to go from the left to the right. So in this place we say a little bit about the representations.

- **:** $(A_l, \bar{\omega}_1)$: the natural representation of A_l, $\Lambda(\bar{\omega}_1) = \{e_1, ..., e_{l+1}\}$
- **:** $(A_l, \bar{\omega}_2)$: the second external power of the natural representation of A_l, the number of weights is given by $C^2_{l+1} = \begin{pmatrix} \text{l+1} \\ 2 \end{pmatrix}$, $\Lambda(\bar{\omega}_2) = \Lambda^2(V(\bar{\omega}_1)) = \{e_i + e_j | 1 \leq i \neq j \leq l+1\}$
- **:** (A_l, ad): the adjoint representation (Cf. Footnote 1). This crystal graph is the diagram of roots of A_l, too. In fact the non-zero weights

[1] Every complex Lie algebra has an unique short root representation, such that its highest weight is the short dominant root of Φ. In case $\Phi = A_l$, $l \geq 1$, where all the roots have the same length, the short root representation is called the adjoint representation. If we have Φ with two length of roots, we have adjoint not basic representation [23].

FIGURE 1. THREE CRYSTAL GRAPHS

(the nodes) are precisely the roots of $\Phi = A_l$, and the bounds are the fundamental roots with respect to the fixed ordering on Φ. If $\alpha = \sum \lambda_i \alpha_i$ is a positive root then it is represented by increasing paths with $|\lambda_1|$ bounds of label 1, ..., $|\lambda_l|$ bounds of label l. The maximal root is at the left winger and at the right winger of (A_l, ad).

1.3. **Quivers.** Here we give a schematic recall of some facts on quivers and their representation theory. Details are, for example, in [1, 8, 25]. We refer in particular, even if with the addition of some our observations, to [8] for the definition of a (Gabriel) quiver and to [1] for the definition of the Auslander-Reiten quiver.
We use something about the theory of categories, that we'll recall in Section 1.4.
For our scope, let K be an algebraically closed field of characteristic zero, say it $\mathbb{C}$. A quiver Γ is a set of vertices connected by arrows. Thus we can write it by $\Gamma = (\Gamma_{\mathcal{V}}, \Gamma_{\mathcal{A}})$ where $\Gamma_{\mathcal{V}}$ is the set of vertices and $\Gamma_{\mathcal{A}}$ is the set of arrows. We denote by $t : \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{V}}$ ($h : \Gamma_{\mathcal{A}} \to \Gamma_{\mathcal{V}}$) the tail (head) map such that any arrow $a \in \Gamma_{\mathcal{A}}$ is mapped onto its tail (head). The case $t(a) = h(a)$ is possible, too: in this case we have a loop. It is obvious that any quiver is pictured by an oriented graph with nodes, that are the vertices in $\Gamma_{\mathcal{V}}$, and

arrows, that are the elements in $\Gamma_{\mathcal{A}}$. Given a quiver, we can delete an its vertex and then all the arrows containing it, and also delete an its arrow and then its tail is identified with its head.
If i, j are two vertices in $\Gamma_{\mathcal{V}}$, then we denote by $\gamma_{ij} = \gamma = a_1|...|a_n$ a path of length n, with $n \geq 1$, from i to j [2] such that $t(a_1) = i$, $t(a_{i+1}) = h(a_i)$ for $1 \leq i \leq n-1$ and $h(a_n) = j$. The composition of two paths is given by the juxtaposition, that is $\underbrace{a_1|...|a_n}_{\gamma_{ij}}\underbrace{b_1|...|b_m}_{\gamma_{hm}} = \underbrace{a_1|...|a_n|b_1|...|b_m}_{\gamma_{im}}$ if $j = h$ and 0 otherwise. A path such that $t(a_1) = h(a_n)$ is called parallel.
The dual quiver of Γ is Γ^*, with $\Gamma_{\mathcal{V}}, \Gamma_{\mathcal{A}}$ as in Γ but with the arrows in $\Gamma_{\mathcal{A}}$ of opposite orientation: if $a \in \Gamma_{\mathcal{V}}$, then a^* is its opposite. There is a bijection $*\colon \Gamma \to \Gamma^*$ such that $*(i) = i$, for all $a \in \Gamma_{\mathcal{V}}$ and $*(a) = a^*$, for all $a \in \Gamma_{\mathcal{A}}$. Given any γ as above, we define similarly $\gamma^* = a_n^*|...|a_1^*$, which is a path of length n, with $n \geq 1$, from j to i.
From Γ, we construct the Category of paths, say $\mathcal{C}_\Gamma$, in which the set of the objects is $\Gamma_{\mathcal{V}}$, and the morphisms are the sets $\mathcal{C}_{\Gamma_{ij}}$ of the paths from i to j, for all $i, j \in \Gamma_{\mathcal{V}}$.The law of composition is the juxtaposition of paths. The linearisation of this category, is the K-category of paths of Γ, that we denote by $K\Gamma$. As vector space, its basis is given by the set of all the paths that we have defined above, and the multiplication is just the juxtaposition. It is a K-algebra, with a unit $1_{K\Gamma}$ if Γ_0 is a finite set. $K\Gamma$ is called a finite dimensional algebra if and only if Γ has finitely many vertices and arrows and it has no (oriented) loops; it is an algebra of finite representation type, that is it has only a finite number of non isomorphic indecomposable modules.
A representation of a quiver Γ over K is a family $V = (V_i, \phi_a)$ such that for $i \in \Gamma_{\mathcal{V}}$, V_i a K-module and for any $a \in \Gamma_{\mathcal{A}}$ such that $t(a) = i$, $h(a) = j$, there exists a linear transformation $\phi_a : V_i \longmapsto V_j$. The two maps $i \to V_i$ and $a \to \phi_a$, extend to a K-functor $\Phi : K\Gamma \to Mod_K$, thus there is a one to one correspondence between the representations of Γ over K and the (left)$K\Gamma$-modules.
Given two quiver representations $(V_i, \phi_a), (V_i', \phi_a')$, there exists a morphism $\varphi : (V_i, \phi_a) \longmapsto (V_i', \phi_a')$ which consists of linear transformations $\varphi_i : V_i \longmapsto V_i'$, $i \in \Gamma_{\mathcal{V}}$, such that for any arrow $a \in \Gamma_{\mathcal{V}}$, with $t(a) = i$ and $h(a) = j$, one has: $\varphi_i(j)\phi_a = \phi_a'\varphi_i(i)$.
Let Γ be a quiver with $\Gamma_{\mathcal{V}} = \{1, 2, ..., l\}$ and V be a representation of Γ over K: its dimension type is given by $\underline{\dim}(V) = (\dim(V_1), ..., \dim(V_l)) \in \mathbb{Q}^n$, and $(\underline{\dim}(V))_i = \dim\,(V_i)$, for $i = 1, ..., l$.
For completeness, since it is not our case, we introduce the notion of a

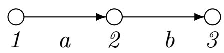
[2]Really in [8], the authors use $\gamma = a_n|...|a_1$ to denote the same path; our notation, is given in sight of our results.

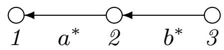
relation ρ for a quiver Γ. Given any $i, j \in \Gamma_{\mathcal{V}}$, ρ is defined as a linear combination of some paths of length at least two, from i to j. Let$< \rho_i | i \in I >$ be the ideal of $K\Gamma$, which is generated by the relations (here i is an index and I is a set of indices). Then the representations of Γ correspond to (left) $K\Gamma / < \rho_i >$-modules.
The smallest significant quiver is given by one vertex with a loop, that is $\Gamma = (\{1\}, a \circlearrowleft)$ with dual $\Gamma^* = (\{1\}, a \circlearrowright)$. We have $K\Gamma \cong K[x]$, the algebra of polynomials in one variable . Suppose $\rho = a^2 - a$, then $K\Gamma / < \rho > \cong K[x]/(a^2 - a)$. In our paper, we refer to the case $l = 1$ (without both loop and relations)only in Section 2, then for us, the smallest significant quiver is $\Gamma = (\{1, 2\}, \{1 \xrightarrow{a} 2\})$ with dual $\Gamma^* = (\{1, 2\}, \{1 \xleftarrow{a^*} 2\})$.

Let A be a finite dimensional algebra over a commutative field. We introduce another quiver: the Auslander-Reiten quiver of $K\Gamma$, ARQ for sake of brevity. In [1], the authors treated the problem of the representations theory of Artin algebras (an Artin algebra is a ring which is finitely generated over its artinian center), as example we have the finite dimensional algebras over a field. Moreover they compared description of ARQ over an Artin algebra, both with the ARQ of an algebra of finite representation type, and with the quiver of the Auslander algebra of an artinian algebra of finite representation type. We adapt the definition to our case.
Let $[M]$ denote the isomorphisms class of the indecomposable module M, AQR is defined as follows: its vertices correspond to $[M]$, and two of them, say $[M], [M']$ are linked by an arrow $f : [M] \longmapsto [M']$ if and only if the map $f : M \longmapsto M'$ is irreducible. We recall that a map f is said to be irreducible if it is neither a split monomorphism or a split epimorphism and if, given any factorization $f = f'f''$ then f' is a split epimorphism or f'' is a split monomorphism.
Sometimes ARQ is drawing by explicit indecomposable representations.

Let's fix some notation: M will be denoted also by P or I with the meaning of projective or injective modules as usual; M^* stands for "the dual of M"; $r(M)$ denotes the radical of M. In particular if Γ is a quiver with a linear orientation, $[P_1]$ denotes the isomorphisms class of the indecomposable module of dimension l and P_1 is the unique, up to isomorphisms, indecomposable module of dimension l. The generic node of AQR is denoted by $[r^i P_1 / r^j P_1]$, where $r^i P_1 / r^j P_1$ is, up to isomorphisms, a module of dimension $j - i$, for $j > i$. Thus $r^i P_1 = P_{i+1}$ is, up to isomorphisms, a module of dimension $l - i$, where $P_1 = r^0 P_1$.

A last remark. In this Section we have written about representations, while in Section 1.2, we have discussed the irreducible representations. This is not a problem: in fact, because of the nature of "our" K, an irreducible representation is also an indecomposable one and conversely.

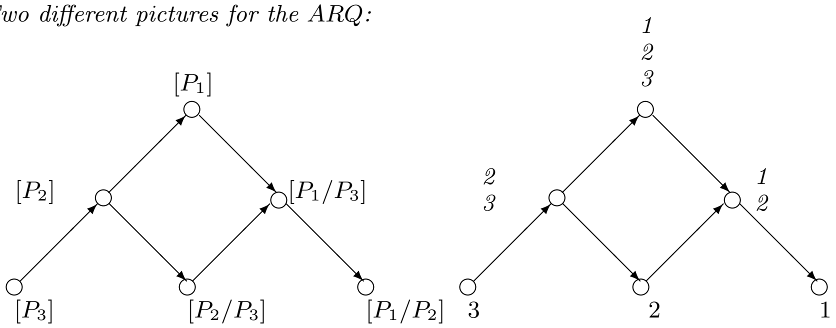
Example 1.1. *Let* $\Gamma = (\{1,2,3\},\{a,b\})$ *be the quiver :* 

and $\Gamma^* = (\{1,2,3\},\{a^*,b^*\})$ *its dual :* 

$K\Gamma$ *is the set of the upper* 3×3 *triangular matrices* $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$ *with entries in* K.
Its basis as vector space (see sec 1.3) is given by the paths $\{a,b,ab\}$. *Its basis over* K *is given by* $B_K(K\Gamma) = \{e_1,e_2,e_3,a,b,ab\}$ *where* e_1,e_2,e_3 *are the idempotents of* $K\Gamma$,*(the idempotent* e_i *is the matrix* $l\times l$ *whose unique non zero entry is* e_{ii} *and equals* 1*).*
If $V = (V_1,V_2,V_3,\phi_a,\phi_b)$ *is a representation of* Γ, *one has:*

$\underline{dim}(V)$	*realization*
(0, 0, 1)	$0 \to 0 \to K$
(0, 1, 0)	$0 \to K \to 0$
(1, 0, 0)	$K \to 0 \to 0$
(0, 1, 1)	$0 \to K \overset{\cong}{\to} K$
(1, 1, 0)	$K \overset{\cong}{\to} K \to 0$
(1, 1, 1)	$K \overset{\cong}{\to} K \overset{\cong}{\to} K$

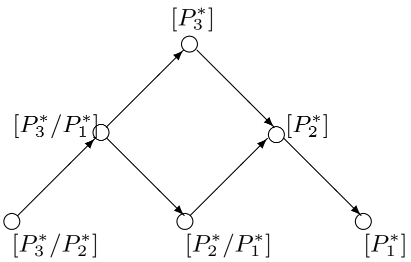
Two different pictures for the ARQ:

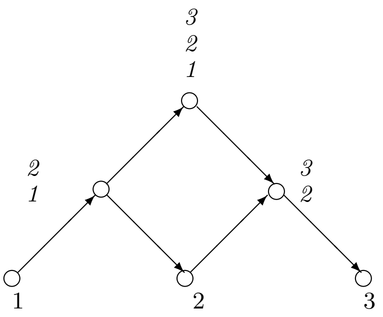


The arrows with direction from bottom to top are the injective morphisms of modules, while the arrows with reverse direction are surjective morphisms of modules.
We read it from bottom to top and collect our results in the following table:

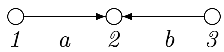
P_i	$dim_K(P_i)$	$P_i \cong A \in M_3(K)$	$B_K(P_i)$	*representation*
$r^2P_1 = P_3$	*1*	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong e_3(K\Gamma)$	$\{e_3\}$	$3{:}0 \to 0 \to K$
$rP_1/r^2P_1 = P_2/P_3$	*1*	$\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} / \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\{e_2\}$	$2{:}0 \to K \to 0$
$P_1/rP_1 = P_1/P_2$	*1*	$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} / \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\{e_1\}$	$1{:}K \to 0 \to 0$
$rP_1 = P_2$	*2*	$\begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong e_2(K\Gamma)$	$\{e_2, b\}$	$\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}{:}0 \to K \overset{\cong}{\to} K$
$P_1/r^2P_1 = P_1/P_3$	*2*	$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} / \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\{e_2, a\}$	$\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}{:}K \overset{\cong}{\to} K \to 0$
P_1	*3*	$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong e_1(K\Gamma)$	$\{e_1, a, ab\}$	$\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}{:}K \overset{\cong}{\to} K \overset{\cong}{\to} K$

$K\Gamma^*$ is the set of the lower 3×3 triangular matrices $\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$ *with entries in K.*
Its basis as vector space (see sec 1.3) is given by the paths $\{a^*, b^*, (ab)^*\}$. Its basis over K is given by $B_K(K\Gamma) = \{e_1, e_2, e_3, a^*, b^*, (ab)^*\}$ where e_1, e_2, e_3 are the idempotents of $K\Gamma$.
Two pictures for ARQ:





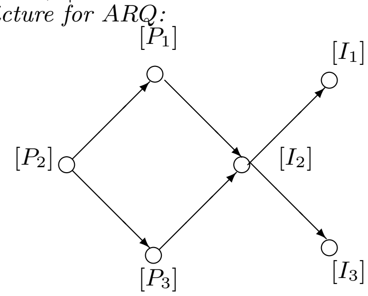
$P_i^*(P_i)$	$dim_K(P_i^*)$	$P_i^* \cong A \in M_3(K)$	$B_K(P_i)$	*representation*
$P_3^*/P_2^*(P_3)$	*1*	$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} / \begin{pmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\{e_3\}$	$3{:}0 \leftarrow 0 \leftarrow K$
$P_2^*/P_1^*(P_2/P_3)$	*1*	$\begin{pmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} / \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\{e_2\}$	$2{:}0 \leftarrow K \leftarrow 0$
$P_1^*(P_1/P_2)$	*1*	$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong e_1(K\Gamma^*)$	$\{e_1\}$	$1{:}K \leftarrow 0 \leftarrow 0$
$P_3^*/P_1^*(P_2)$	*2*	$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} / \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\{e_3, b^*\}$	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}{:}0 \leftarrow K \overset{\cong}{\leftarrow} K$
$P_2^*(P_1/P_3)$	*2*	$\begin{pmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong e_2(K\Gamma^*)$	$\{e_2, a^*\}$	$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}{:}K \overset{\cong}{\leftarrow} K \leftarrow 0$
$P_3^*(P_1)$	*3*	$\begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong e_3(K\Gamma^*)$	$\{e_3, b^*, (ab)^*\}$	$\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}{:}K \overset{\cong}{\leftarrow} K \overset{\cong}{\leftarrow} K$

Example 1.2. *Let* $\Gamma = (\{1,2,3\},\{a,b\})$ *be the quiver:* 

$K\Gamma$ is the set of the matrices of the type $\begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{pmatrix}$ *with entries in K. Its basis as vector space (see sec. 1.3) is given by the paths $\{a,b\}$. Its basis over K is given by $B_K(K\Gamma) = \{e_1, e_2, a, b, ab\}$, where e_1, e_2 are the idempotents of $K\Gamma$.*
If $V = (V_1, V_2, V_3, \phi_a, \phi_b)$ is a representation of Γ, one has:

$\underline{dim}(V)$	*realization*
(0, 0, 1)	$0 \to 0 \leftarrow K$
(1, 0, 0)	$K \to 0 \leftarrow 0$
(0, 1, 0)	$0 \to K \leftarrow 0$
(0, 1, 1)	$0 \to K \overset{\cong}{\leftarrow} K$
(1, 0, 1)	$K \to 0 \leftarrow K$
(1, 1, 1)	$K \overset{\cong}{\to} K \overset{\cong}{\leftarrow} K$

A picture for ARQ:



We read it and collect some results in the following table:

M	$dim_K(M)$	*representation*
P_2	1	$2 : 0 \to 0 \leftarrow K$
$I_1 \cong P_2/P_1$	1	$1 : K \to 0 \leftarrow 0$
$I_3 \cong P_3/P_1$	1	$3 : 0 \to K \leftarrow 0$
P_3	2	$\begin{smallmatrix} 3 \\ 2 \end{smallmatrix} : 0 \to K \overset{\cong}{\leftarrow} K$
P_1	2	$\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} : K \to K \leftarrow 0$
I_2	3	$\begin{smallmatrix} 1 \\ 2\,3 \end{smallmatrix} : K \overset{\cong}{\to} K \overset{\cong}{\leftarrow} K$

Example 1.3. *Let Γ^* be the dual of Γ in Example 1.2.The properties of ARQ are obvious*

After the above examples, if Γ is a quiver of type A_l, $l \geq 4$ with any orientation, the reader is able to find the properties of ARQ by pictures.

1.4. **Categories and functors.** We recall only the essential definitions. See for example [19] for details.
A category $\mathcal{C}$ is given by a class of objects $C_1, C_2, ...$ and a set $M_{\mathcal{C}}$ =

$M_{\mathcal{C}}(C_i, C_j) = M(C_i, C_j)$ of morphisms from C_i to C_j, for all $C_i, C_j \in \mathcal{C}$. The law of composition l is given by $M(C_1, C_2) \times M(C_2, C_3) \to M(C_1, C_3)$ holds such that $M(C_1, C_2)$ and $M(C_3, C_4)$ are disjoint unless $C_1 = C_3, C_2 = C_4$.
We have the following axioms

(1) Associativity of l:
if $\varphi : C_1 \to C_2, \psi : C_2 \to C_3$ and $\chi : C_3 \to C_4$ are given, then $\varphi(\psi\chi) = (\varphi\psi)\chi$ holds;
(2) Existence of the identity:
to each $C \in \mathcal{C}$, there is $1_C : C \to C$, such that for any $\varphi : C_1 \to C_2$, $\psi : C_3 \to C_1$ one has $\varphi 1_C = \varphi$ and $1_C \psi = \psi$.
The morphism 1_C is uniquely dterminated.

A (Covariant) functor F is a map from a category $\mathcal{C}$ to a category $\mathcal{D}$ such that with every object $C \in \mathcal{C}$ there is an object $F(C) = D \in \mathcal{D}$ and that $M_{\mathcal{C}}(C_i, C_j) \to M_{\mathcal{D}}(F(C_i), F(C_j))$. In particular the functor identity $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ such that $1_{\mathcal{C}}(C) = C$ and $1_{\mathcal{C}}(\varphi) = \varphi$, for each $C \in \mathcal{C}$, $\varphi \in M_{\mathcal{C}}(C_i, C_j)$.
Consider two functors $F, G : \mathcal{C} \to \mathcal{D}$ and $\varphi \in Mor_{\mathcal{C}}(C_i, C_j)$.
A natural transformation or a morphism of functors is a mapping $\psi : F \to G$ defined as follows:
Suppose that for every $C \in \mathcal{C}$ there is a $\psi_C : F(C) \to G(C)$such that for any $\varphi \in Mor_{\mathcal{C}}(C_i, C_j)$ the diagram below is commutative:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(\varphi)} & F(C_1) \\ \psi_C \downarrow & & \downarrow \psi_{C_1} \\ G(C) & \xrightarrow[G(\varphi)]{} & G(C_1) \end{array}$$

If there exists also a natural transformation $\eta : G \to F$such that $\psi\eta = 1_G$ and $\eta\psi = 1_F$, then ψ is called a natural equivalence or an isomorphism of functors. We'll write $F \sim G$. Here $1_F : F \to F$ is such that $\eta_C \psi_C = 1_{F(C)}$.
Given any two categories $\mathcal{C}$ and $\mathcal{D}$ and $\mathcal{C} \overset{F}{\underset{G}{\rightleftarrows}} \mathcal{D}$ such that $GF \sim 1_{\mathcal{C}}$ and $FG \sim 1_{\mathcal{D}}$, then we say that $\mathcal{C}$ and $\mathcal{D}$ are equivalent categories $\mathcal{C} \sim \mathcal{D}$.
A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is a category such that its objects are objects in $\mathcal{C}$ too and $M_{\mathcal{C}'} \subset M_{\mathcal{C}}$. If $M_{\mathcal{C}'} = M_{\mathcal{C}}$, then $\mathcal{C}'$ is called a full subcategory of $\mathcal{C}$.
$\varphi \in Mor_{\mathcal{C}}(C_i, C_j)$ is a monomorphism if, given $\bar{\varphi} : \bar{C} \to C_i$ and $\bar{\bar{\varphi}} : \bar{C} \to C_i$ such that $\varphi\bar{\varphi} = \varphi\bar{\bar{\varphi}}$, then $\bar{\varphi} = \bar{\bar{\varphi}}$
$\varphi \in Mor_{\mathcal{C}}(C_i, C_j)$ is a epimorphism if, given $\bar{\varphi} : C_j \to \bar{C}$ and $\bar{\bar{\varphi}} : C_j \to \bar{C}$ such that $\bar{\varphi}\varphi = \bar{\bar{\varphi}}\varphi$, then $\bar{\varphi} = \bar{\bar{\varphi}}$
$\varphi \in Mor_{\mathcal{C}}(C_i, C_j)$ is an isomorphism if there exists a $\bar{\varphi} \in Mor_{\mathcal{C}}(C_j, C_i)$ such that $\bar{\varphi}\varphi = 1_{C_1}$ and $\varphi\bar{\varphi} = 1_{C_j}$. In what follows $\mathcal{C}(A, \varphi)$ stands for a

family of objects A and a generic morphism φ between objects in $\mathcal{C}$. In particular, we consider $\mathcal{C}_{ARQ_\Phi}([M], f)$, where $[M]$ are defined in Section 1.3, and the sub-subscript $_\Phi$ stress what is the underlying Dynkin diagram; $\mathcal{C}_{\Phi,\bar{\omega}_k}(\Lambda(\bar{\omega}_k), i)$, where $\Lambda(\bar{\omega}_k)$ and the edges i of the crystal graphs are defined in Section 1.2,
$\mathcal{C}_{\Phi,\bar{\omega}_k}(\lambda, i^*)$, where λ is a weight in $\Lambda(\bar{\omega}_k)$ and $* = \pm 1$ as they'll be defined in Remark 3.21

2. A first result

Let (A_l, ad) be the adjoint representation, and with a little abuse of notation, we indicate its crystal graph by the same (A_l, ad). As it looks in Figure 1, it is symmetric with respect to a vertical axis through the zero weights; we indicate by $(A_l, ad)_{Sym}$ its left part without the zero-weights and relative edges.
In this section (A_l, ad) is the root diagram (see Section 1.2) and describe any root $\alpha \in \Phi$ with $ht(\alpha) = k$, $1 \le k \le l$, as an array of length l with unique subsequently no zero k entries equal to 1. We observe that
$(0, ..., 0, \underbrace{1, ..., 1}_{l-i}) - (0, ..., 0, \underbrace{1, ..., 1}_{l-j}) = (0, ..., 0, \underbrace{1, ..., 1}_{j-i}, 0..., 0)$,
for $j \ge i$, $i = 0, ..., l-i$. The first result is another proof $(ii)_G$ of the Introduction.
In what follows we work directly on the coset representatives.

Definition 2.1. *Let $l \ge 1$ be a fixed natural number. Let L be a semi-simple complex Lie algebra of type A_l, and Γ be a quiver of type A_l with linear orientation* $\underset{1}{\circ} \longrightarrow \underset{2}{\circ} \cdots \underset{l-1}{\circ} \longrightarrow \underset{l}{\circ}$
Let $K\Gamma$ be the path algebra of Γ. Then there is a map η from $(A_l, ad)_{Sym}$ to ARQ such that:
$\eta(\alpha) = \eta(\sum_{k=i+1, \alpha_k \in \Pi}^{j} \alpha_k) = r^i P_1 / r^j P_1$, *for* $j > i, i = 0, ..., l-1, j = 1, ..., l-1$.

Lemma 2.1. *We refer to Definition 2.1. The map η is an isomorphism from $(A_l, ad)_{Sym}$ to ARQ.*
It is also a isomorphism between the edges and the morphisms, respectively.

Proof The case $l = 1$ is obvious, in fact $\Phi^+(A_1) = \alpha = \tilde{\alpha}$, (A_1, ad): $\circ\!\!-\!\!\circ\!\!-\!\!\circ$.
Then there exists a unique indecomposable (simple) module P which corresponds to the unique root α.
From now on l is at least 2.

Let $\alpha = \tilde{\alpha}$ with array $(\underbrace{1, 1, ..., 1}_{l})$. Then $\eta(\alpha) = P_1$, since $\dim_K(P_1) = l$ and it corresponds to the representation V of Γ with $\underline{\dim}(V) = (\underbrace{1, 1, ..., 1}_{l})$.

Let $\alpha = \sum_{h=1}^{i} 0\alpha_h + \sum_{h=i+1}^{l} 1\alpha_h$ with array $(0, ..., 0, \underbrace{1, ..., 1}_{l-i})$. Then $\eta(\alpha) = r^i P_1$ since $\dim_K(r^i P_1) = l - i$ and it corresponds to the representation V of Γ with $\underline{\dim}(V) = (0, ..., 0, \underbrace{1, ..., 1}_{l-i})$.

Let $\alpha = \sum_{h=1}^{i} 0\alpha_h + \sum_{h=i+1}^{l} 1\alpha_h - (\sum_{h=1}^{j} 0\alpha_h + \sum_{h=j+1}^{l} 1\alpha_h) = \sum_{h=1}^{i} 0\alpha_h + \sum_{h=i+1}^{j} 1\alpha_h + \sum_{h=j+1}^{l} 0\alpha_h$ with $(0, ..., 0, \underbrace{1, ..., 1}_{j-i}, 0..., 0)$. Then $\eta(\alpha) = r^i P_1 / r^j P_1$, since $\dim_K(r^i P_1 / r^j P_1) = j - i$ and it corresponds to the representation V of Γ with $\underline{\dim}(V) = (0, ..., 0, \underbrace{1, ..., 1}_{j-i}, 0..., 0)$.

η is injective. In fact, let $\bar{\alpha}, \alpha \in \Phi$ be any two different roots $\bar{\alpha}, \alpha \in \Phi$. At first we suppose that $ht(\bar{\alpha}) \neq ht(\alpha)$: we have $\eta(\bar{\alpha}) \neq \eta(\alpha)$. At the contrary, if $ht(\bar{\alpha}) = ht(\alpha)$, then their arrays are different and then we have $\eta(\bar{\alpha}) \neq \eta(\alpha)$. η is surjective, it is a simple exercise.
It follows that $\eta(i) = f_i$ where if i stands for $\alpha_i \in \Pi$ and $i : \alpha \to \bar{\alpha} \in (A_l, ad)_{Sym}$, then $f_i : \eta(\alpha) \to \eta(\bar{\alpha}) \in ARQ$. $\square$

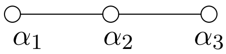
Remarks 2.1. *We refer to Section 1.4 and have* $\mathcal{C}_{ARQ_{A_l}}([P_i], f)$ *and* $\mathcal{C}_{(A_l,ad)_{Sym}}(\beta_i, \alpha) = \mathcal{C}_{(A_l,ad)}(\beta_i, \alpha)$*, where* $\beta_i \in \Phi = A_l$*,* $\alpha \in \Pi$ *is the edge of the representation.*
The map η *in Lemma 2.1 is exactly the functor* $F : \mathcal{C}_{(A_l,ad)_{Sym}}(\beta_i, \alpha) \to \mathcal{C}_{ARQ_{A_l}}([P_i], f)$ *such that* $F(\beta_i) = [P_i]$*, with* $dim_K(P_i) = ht(\beta_i)$ *and* $F(\alpha : \beta_i \to \beta_j) = F(\alpha) : F(\beta_i) \to F(\beta_j)$*.*
Moreover $\mathcal{C}_{ARQ_{A_l}}([P_i], f) \sim \mathcal{C}_{(A_l,ad)_{Sym}}(\beta_i, \alpha)$*.*

Corollary 2.1 ((C) for A_l)**.** *Let* $l \geq 1$ *be a fixed natural number. If the hypothesis of Lemma 2.1 hold, and if* α *is a positive root of* L *then there exists an unique indecomposable representation* V *of* $K\Gamma$ *such that* $dim(V) = \alpha$*.*

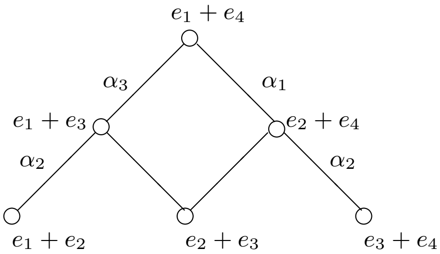
Proof By Lemma 2.1, each node of $(A_l, ad)_{Sym}$ is a root α of L which corresponds to an indecomposable module of dimension equal to $ht(\alpha)$. $\square$

3. **Oriented Dynkin diagrams and oriented crystal graphs**

Let's fix $l = 3$, $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$, D_{A_3}:



$(A_3, \bar{\omega}_2)$ the second fundamental representation of L, where $\bar{\omega}_2 = e_1 + e_2$, $\Lambda(\bar{\omega}_2) = \{e_1+e_2, e_1+e_3, e_1+e_4, e_2+e_3, e_2+e_4, e_3+e_4\}$ and crystal graph:



We observe that only for our convenience, we picture the first and the last edges diagonally.

Definition 3.1 ([20])**.** *Let D_{A_l} be a Dynkin diagram for $l \geq 2$ and let E be the set of its edges. An orientation on A_l is a pair of maps $\sigma, \eta : E \longmapsto A_l$ such that any edge $e \in E$ can be identified with its source $\sigma(e)$ and its end $\eta(e)$.*

We adapt Definition 3.1 to our interest and introduce new notations

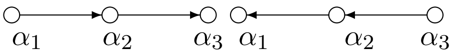
Definition 3.2. **:** $l = 2$

$D_{A_2^{\rightarrow}}$ and $D_{A_2^{\leftarrow}}$:

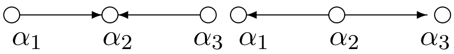
denote the only two linear oriented Dynkin diagrams which are pairwise dual

: $l = 3$

$D_{A_3^{\rightarrow}}$ and $D_{A_3^{\leftarrow}}$ with linear orientations:



$D_{A_3^{\rightarrow\leftarrow}}$ and $D_{A_3^{\leftarrow\rightarrow}}$ with non linear orientations:



denote the only four linear oriented Dynkin diagrams which are pairwise dual

: $l \geq 2$

By $D_{A_l^{\overleftarrow{\rightarrow}}}$ we denote an oriented Dynkin diagram of type A_l with any *non linear orientation on it and by $D_{A_l^{\rightarrow}}$ or $D_{A_l^{\leftarrow}}$ we denote a linear oriented , as in the arrow, Dynkin diagram of type A_l.*

Definition 3.3 ([4]). *Let τ be a permutation of on Φ then:*
(i) the number of roots of the Dynkin diagram is the same
(ii) if i is joined to j then $\tau(i)$ is joined to $\tau(j)$.

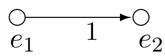
Other new notations:
$\odot$ stands for "gluing";
if $(\Phi, \bar{\omega}_i)$ denotes a representation, then $(\Phi^{\rightarrow}, \bar{\omega}_i)$ denotes an oriented crystal graph or an oriented weight graph.

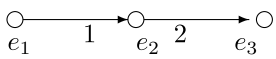
Now we adapt the construction of the weights diagrams of the representations $(A_l, \bar{\omega}_k)$ given by Plotkin, Semenov and Vavilov [23],to the case $k = 2$.

Construction 3.1. *The representation $(A_l, \bar{\omega}_2)$ is constructed by an inductive procedure as follows. Let us make the following identifications: $A_{l-1} = A_l - \{\alpha_l\}$ and $A_{l-2} = A_l - \{\alpha_l, \alpha_{l-1}\}$.*
The weights diagrams of the two representations $(A_{l-1}, \bar{\omega}_2)$ and $(A_{l-1}, \bar{\omega}_1)$ are glued together along the representation $(A_{l-2}, \bar{\omega}_1)$ with label α_l, that is $(A_l, \bar{\omega}_2) = (A_{l-1}, \bar{\omega}_2) \odot_{(A_{l-2}, \bar{\omega}_1)_l} (A_{l-1}, \bar{\omega}_1)$.

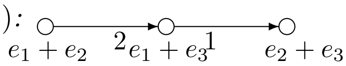
Next we make a new construction to obtain oriented crystal graphs and new notations.We start with

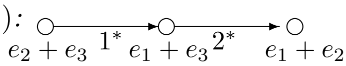
Remarks 3.1. *Our primary request about the oriented crystal graphs, is that the underlying structures are crystal graphs. For this we want that the edges are fundamental roots and that our pictures start from $\bar{\omega}_2$. Let's comment two preliminary cases:*

: *construct $(A_l^{\rightarrow}, \bar{\omega}_1)$.*
Consider the crystal graph of the representation $(A_l, \bar{\omega}_1)$, $l \geq 1$, they are chains. Thus starting from $\bar{\omega}_1$ only an orientation is possible; for example, for $l = 1$ and $l = 2$ we have $(A_1^{\rightarrow}, \bar{\omega}_1)$: 

and $(A_2^{\rightarrow}, \bar{\omega}_1)$: 

: *construct $(A_2^{\rightarrow}, \bar{\omega}_2)$ and $(A_2^{\leftarrow}, \bar{\omega}_2)$*
Consider the crystal graph of the representation $(A_2, \bar{\omega}_2)$ it is a chain. Thus starting from $\bar{\omega}_2$ only an orientation is possible;

$(A_2^{\rightarrow}, \bar{\omega}_2)$: 

$(A_2^{\leftarrow}, \bar{\omega}_2)$: 

Here we indicate the oriented edges as if they would be quivers, so i^* denotes the edge with opposite orientation with respect to the edge i.

Construction 3.2. *The oriented weight diagrams $(A_l^{\rightarrow}, \bar{\omega}_k)$ $((A_l^{\leftarrow}, \bar{\omega}_k))$ with respect to $D_{A_l^{\rightarrow}}$ $(D_{A_l^{\leftarrow}})$, for $l \geq 3$ and $\bar{\omega}_k, k = 1, 2$, are constructed by an inductive procedure. Then we obtain*

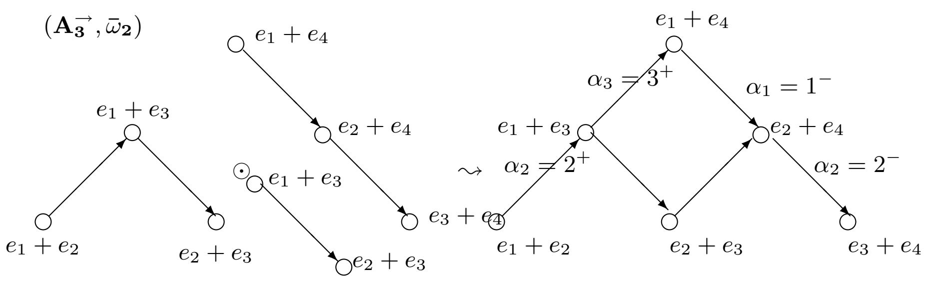
$$(A_3^{\rightarrow}, \bar{\omega}_2) = (A_2^{\rightarrow}, \bar{\omega}_2) \odot_{(A_1^{\rightarrow}, \bar{\omega}_1)_3^+} (A_2^{\rightarrow}, \bar{\omega}_1) \tag{1}$$

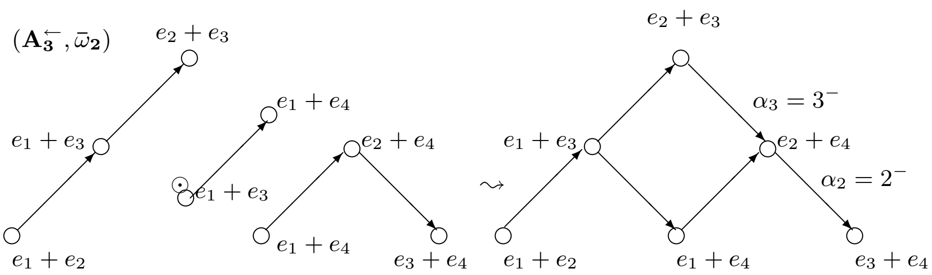
$$(A_3^{\leftarrow}, \bar{\omega}_2) = (A_2^{\leftarrow}, \bar{\omega}_2) \odot_{(A_1^{\rightarrow}, \bar{\omega}_1)_3^-} (A_2^{\rightarrow}, \bar{\omega}_1) \tag{2}$$

$$(A_l^{\rightarrow}, \bar{\omega}_2) = (A_{l-1}^{\rightarrow}, \bar{\omega}_2) \odot_{(A_{l-2}^{\rightarrow}, \bar{\omega}_1)_{l^+}} (A_{l-1}^{\rightarrow}, \bar{\omega}_1) \tag{3}$$

$$(A_l^{\leftarrow}, \bar{\omega}_2) = (A_{l-1}^{\leftarrow}, \bar{\omega}_2) \odot_{(A_{l-2}^{\rightarrow}, \bar{\omega}_1)_{l^-}} (A_{l-1}^{\rightarrow}, \bar{\omega}_1) \tag{4}$$

Proof Consider $D_{A_3^{\rightarrow}}$ and pone $A_2 = A_3 - \{\alpha_3\}$, $A_1 = \{\alpha_1\}$ as suggested in Construction 3.1, thus one has:
$(A_2^{\rightarrow}, \bar{\omega}_2)$, with $\bar{\omega}_2 = e_1 + e_2$, $(A_2^{\rightarrow}, \bar{\omega}_1)$, with $\bar{\omega}_1 = e_1$ (really we have the weights $e_1 + e_4$, $e_2 + e_4$, $e_3 + e_4$ which constitute a set of weight in $V(\bar{\omega}_1)$), and $(A_1^{\rightarrow}, \bar{\omega}_1)$, with $\bar{\omega}_1 = e_1 + e_3$ (really we have the weights $e_1 + e_3$, $e_2 + e_3$ which constitute a set of weight in $V(\bar{\omega}_1)$). By gluing we obtain (1) where 3^+ stands for an oriented edge of type $\nearrow$ with label 3.
We can use a similar procedure to obtain $(A_3^{\leftarrow}, \bar{\omega}_2)$.
Consider $D_{A_3^{\leftarrow}}$. From Definition 3.3, this means substantially that we permute the roots in Φ. In fact we pone $A_2 = A_3 - \{\alpha_3\}$, $A_1 = \{\alpha_1\}$, as suggested in Construction 3.1, thus one has:
$(A_2^{\leftarrow}, \bar{\omega}_2)$, with $\bar{\omega}_2 = e_2 + e_3$, $(A_2^{\rightarrow}, \bar{\omega}_1)$, with $\bar{\omega}_1 = e_1$ (really we have the weights $e_1 + e_4$, $e_2 + e_4$, $e_3 + e_4$ which constitute a set of weight in $V(\bar{\omega}_1)$) and $(A_1^{\rightarrow}, \bar{\omega}_1)$, with $\bar{\omega}_1 = e_1 + e_4$ (really we have the weights $e_1 + e_4$, $e_2 + e_4$ which constitute a set of weight in $V(\bar{\omega}_1)$). By gluing we obtain (2) where 3^- stands for an oriented edge of type $\searrow$ with label 3.
By induction we obtain (3), (4).
We end the proof with two picture for the case $l = 3$.





□

We have learnt to draw the oriented crystal graphs of type $(A_l^{\rightarrow}, \bar{\omega}_2)$, but what about their descriptions? All is in the Remarks below.

Remarks 3.2. *These Remarks are about Construction 3.2.*

(1) *We can repeat all said in Section 1.2 for the pictures and the reading of the representations even for our oriented crystal graphs, but here we have to be careful: any oriented edge can be either of type* $\nearrow$ *(that is a proper inclusion with images of codimension one) or of type* $\searrow$ *(that is a proper quotient with kernel of dimension one). So when the labels* α_l *mean oriented maps, we denote them by* l^+ *or by* l^-*. For this reason in our pictures above, we have used both the notations;*

(2) *Consider* $(A_3^{\rightarrow}, \bar{\omega}_2)$*. We can read on it two paths: one with the vertices* $e_1 + e_2, e_1 + e_3, e_1 + e_4, e_2 + e_4, e_3 + e_4$ *corresponding to orientation "$\rightarrow$" and the other one with vertices* $e_1 + e_2, e_1 + e_3, e_2 + e_3, e_2 + e_4, e_3 + e_4$ *corresponding to the same orientation after the permutations of the roots.*
Look at the frameworks of these two oriented crystal graphs: in the first case, we have the weight graph of the second external representation $(A_3, \bar{\omega}_2)$*; in the second case we have the weight graph of the contragradient representation with respect to* $(A_3, \bar{\omega}_2)$*. (In the contragradient representation the module corresponding to* $\bar{\omega}_k$*, is given by* $V^*(\bar{\omega}_k) = V(\bar{\omega}_{l+1-k})$*).*

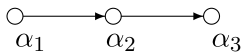
In the sequel, we'll make use of the following standard notations:
$\mathcal{V}(e_i)$ is the vector space over $\mathbb{C}$ generated by e_i;
$B_K(V)$ is the basis of the K-vector space V;
V_{P_i} is the module P_i as a vector space.

Remarks 3.3. *Our claim is to prove that some finite-dimensional vector spaces, which are generated by weights and fundamental roots are the images of certain indecomposable modules (as vector spaces) of AQR. The source of this is the choice to put together the nodes of our oriented crystal graphs, which are the weights of the representation as usual, with finite dimensional vector spaces. The procedure is like-natural. In fact we can move from a weight to another by the action of Weyl group (see Section 1.2). So that, if $\lambda - \mu = \alpha_i$, we identify the node with $\mathcal{V}(\lambda, \alpha_i)$.*

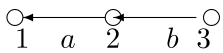
Definition 3.4. *Let L be a semi-simple complex Lie Algebra of type A_3 and $D_{A_3^{\rightarrow}}$ ($D_{A_3^{\leftarrow}}$) be its linear oriented Dynkin diagram. Let $K\Gamma$ be the path algebra of Γ, a quiver of type A_3 with the same linear orientation than D_{A_3}. Then there is a map $\eta = \eta_{A_3^{\rightarrow}}$ ($\eta_{A_3^{\leftarrow}}$) from ARQ to $(A_3^{\rightarrow}, \bar{\omega}_2)$, $((A_3^{\leftarrow}, \bar{\omega}_2))$, which is defined as follows. Take the class $[P_3]$ which is at the bottom and on the left of AQR. We pone $\eta([P_3]) = \mathcal{V}(e_1 + e_2)$. Next, looking at the arrows of AQR we have $\eta([M]) = \mathcal{V}(e_1 + e_2, \alpha_1, ..., \alpha_h)$ with $\alpha_1, ..., \alpha_h \in \Pi$, $h = 0, 2, 3$, such that $dim(V)_M = dim\mathcal{V}(e_1 + e_2, \alpha_1, ..., \alpha_h)$, or a quotient of vector spaces of this type.*

Lemma 3.1. *We refer to definition 3.4. The map $\eta = \eta_{A_3^{\rightarrow}}$ ($\eta_{A_3^{\leftarrow}}$) is an isomorphism, from ARQ to $(A_3^{\rightarrow}, \bar{\omega}_2)$, $((A_3^{\leftarrow}, \bar{\omega}_2))$. This map is also an isomorphism between oriented edges of $(A_3^{\rightarrow}, \bar{\omega}_2)$, $((A_3^{\leftarrow}, \bar{\omega}_2))$ and the morphisms in ARQ, in an obvious way. Moreover $\eta|_{A_2^{\rightarrow} = A_3^{\rightarrow} - \{\alpha_3\}}$ is an isomorphism in the sense above, too.*

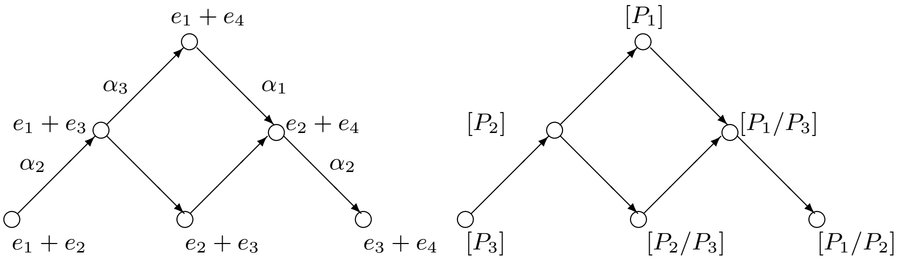
Proof Consider $D_{A_3^{\rightarrow}}$:



Γ:



$(A_3^{\rightarrow}, \bar{\omega}_2)$, ARQ:



In what follows we consider for each $[M]$ in ARQ a coset representative. The map η is obviously well defined . We explicit it.
We pone $\eta(P_3) = \mathcal{V}(e_1 + e_2)$, in fact The indecomposable module $P_3 = rP_2 = r(r(P_1))$ is such that $\dim_K(P_3) = 1$ and $B_K(P_3) = \{e_3\}$. Thus $\dim_{\mathbb{K}}(V_{P_3}) = 1$.

The indecomposable module P_2 is such that $\dim_K(P_2) = 2$ and $B_K(P_2) = \{e_3, b\}$. Thus $\dim_{\mathbb{K}}(V_{(P_2)} = 2$. One has $\eta(P_2) = \mathcal{V}(e_1 + e_2, \alpha_2)$.
The indecomposable module P_1 is such that $\dim_K(P_1) = 3$ and $B_K(P_1) = \{e_3, b, ab\}$. Thus $\dim_{\mathbb{K}}(V_{P_1}) = 3$. One has $\eta(P_1) = \mathcal{V}(e_1 + e_2, \alpha_2, \alpha_3)$.
The indecomposable module P_1/P_3 is such that $\dim_K(P_1/P_3) = 2$ and $B_K(P_1/P_3) = \{e_2, a\}$.Thus $\dim_{\mathbb{C}}(P_1/P_3) = 2$. One has $\eta(P_1/P_3) = \mathcal{V}(e_1 + e_2, \alpha_2, \alpha_3)/\mathcal{V}(e_1 + e_2)$;
The indecomposable module P_1/P_2 is such that $\dim_K(P_1/P_2) = 1$ and $B_K(P_1/P_2) = \{e_1\}$. Thus $\dim_{\mathbb{K}}(P_1/P_2) = 1$. One has $\eta(P_1/P_2) = \mathcal{V}(e_1 + e_2, \alpha_2, \alpha_3)/\mathcal{V}(e_1 + e_2, \alpha_1)$;
The indecomposable module P_2/P_3 is such that $\dim_K(P_2/P_3) = 1$ and $B_K(P_2/P_3) = \{e_2\}$. Thus $\dim_{\mathbb{C}}(P_2/P_3) = 1$. One has $\eta(P_2/P_3) = \mathcal{V}(e_1 + e_2, \alpha_2)/\mathcal{V}(e_1 + e_2\}$.
The correspondence between maps is obvious. In fact the injective morphisms in ARQ are the proper inclusion maps; the surjective morphisms in ARQ are the proper quotient maps. We can explicit this as follows: $\eta(2^+) = [P_3] \xrightarrow{f} [P_2]$, $\eta(2^-) = [P_1/P_3] \xrightarrow{f} [P_1/P_2]$, and so on, adopting the $l^\pm$ labels.
The proof that η is injective and surjective is a very easy exercise for undergraduate readers.
At last, there is a correspondence between $D_{A_3^{\rightarrow}-\{\alpha_3\}}$ and ARQ of the sub-quiver $(\{1, 2\}, \{1 \overset{a}{\leftarrow} 2\})$.
The proof with respect to the opposite orientation is similar.□

Definition 3.5. *Let L be a semi-simple complex Lie Algebra of type A_l and $D_{A_l^{\rightarrow}}$ ($D_{A_l^{\leftarrow}}$) be its linear oriented Dynkin diagram. Let $K\Gamma$ be the path algebra of Γ, a quiver of type A_l with the same linear orientation than D_{A_l}. Then there is a map $\eta = \eta_{A_l^{\rightarrow}}$ ($\eta_{A_l^{\leftarrow}}$) from ARQ to $(A_l^{\rightarrow}, \bar{\omega}_2)$, $((A_l^{\leftarrow}, \bar{\omega}_2))$. The map is defined as follows. Take the class $[P_l]$ which is at the bottom and on the left of AQR. We pone $\eta([P_l]) = \mathcal{V}(e_1 + e_2)$. Next, looking at the arrows of AQR we have $\eta([M]) = \mathcal{V}(e_1 + e_2, \alpha_1, ..., \alpha_h)$ with $\alpha_1, ..., \alpha_h \in \Pi$, $h = 0, 2, 3, ..., l-1$, such that $dim(V)_M = dim\mathcal{V}(e_1 + e_2, \alpha_1, ..., \alpha_h)$, or a quotient of vector spaces of this type.*

Theorem 3.1. *We refer to definition 3.5. The map $\eta = \eta_{A_l^{\rightarrow}}$ ($\eta_{A_l^{\leftarrow}}$) from ARQ to $(A_l^{\rightarrow}, \bar{\omega}_2)$, $((A_l^{\leftarrow}, \bar{\omega}_2))$ is an isomorphism. Pone $A_{l-1} = A_l - \{\alpha_l\}$, $A_{l-2} = A_l - \{\alpha_{l-1}, \alpha_l\}$, ..., $A_2 = A_l - \{\alpha_3, ...\alpha_l\}$. Thus $\eta|_{A_k^{\rightarrow}}$ is an isomorphism in the sense above, for $k = 2, 3, ..., l-1$, too.*

Proof If $l = 2$ the theorem is obvious. For $l = 3$ Lemma 3.1 holds. Then the result follows by induction on l. □

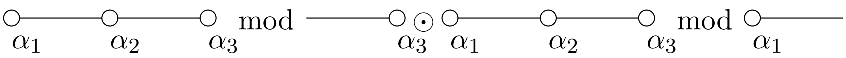
Remarks 3.4. *We consider (Cf. Section 1.4) the following categories:* $\mathcal{C}_{ARQ,A_l}([M],f)$ *and* $\mathcal{C}_{(A_l^{\leftarrow},\bar\omega_2)}(\mathcal{V}(?),l^?)$*, where* $\mathcal{V}(?)$ *stands for a finite dimensional vector space generated by roots and weights, and* $l^?$ *stands for an orientated edge as in Construction 3.2.*
Then the map η *in Theorem 3.1 is exactly the functor*
$F:\mathcal{C}_{ARQ,A_l}([M],f)\to\mathcal{C}_{(A_l^{\leftarrow},\bar\omega_2)}(\mathcal{V}(?),l^?)$ *such that*
$F([M])=\mathcal{V}(?)$ *with* $dim_{\mathbb{C}}(\mathcal{V}(?))=dim_{\mathbb{C}}(V_M)$*. Moreover,* $F([M_1]\stackrel{\varphi}{\to}[M_2])=F([M_1])\stackrel{l^+}{\to}F([M_2])$ *if* φ *is an injective morphism and*
and $F([M_1]\stackrel{\varphi}{\to}[M_2])=F([M_1])\stackrel{l^-}{\to}F([M_2])$ *if* φ *is a surjective morphism.*
We have $\mathcal{C}_{ARQ,A_l}([M],f)\sim\mathcal{C}_{(A_l^{\leftarrow},\bar\omega_2)}(\mathcal{V}(?),l^?)$ *and*
$\mathcal{C}_{ARQ,A_k}([M],f)\sim\mathcal{C}_{(A_k^{\leftarrow},\bar\omega_2)}(\mathcal{V}(?),l^?)$*, for* $k=2,3,...,l-1$*.*

4. A particular weight graph

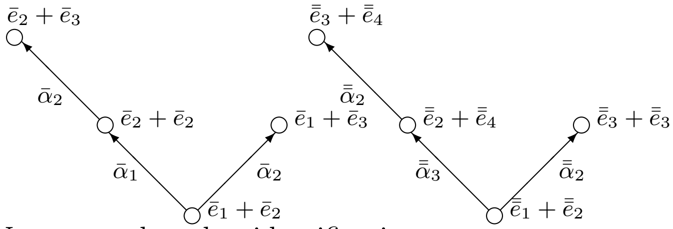
First our aim in this section is to define $(A_3^{\rightarrow\leftarrow},\bar\omega_2)$ as the " gluing" of $(A_3^{\rightarrow},\bar\omega_2)$ and $(A_3^{\leftarrow},\bar\omega_2)$, after some "mod" operations.

Construction 4.1. *The weight graph* $(A_3^{\rightarrow\leftarrow},\bar\omega_2)$ *is constructed by the following identifications. We say that mod*(α_3) *means* $\alpha_3=-\alpha_2$*, then* $e_3-e_4=e_3-e_2$*, and*
mod(α_1) *means* $\alpha_1=-\alpha_2$*, then* $e_1-e_2=e_3-e_2$*.*
Then one has $(A_3^{\rightarrow\leftarrow},\bar\omega_2)=(A_3^{\rightarrow},\bar\omega_2)mod(\alpha_3)\odot(A_3^{\leftarrow},\bar\omega_2)mod(\alpha_1)$*.*

Proof At first we can visualize $''\mathrm{mod}(\alpha_3)''$ and $''\mathrm{mod}(\alpha_1)''$ on D_{A_3} by the aid of a picture:

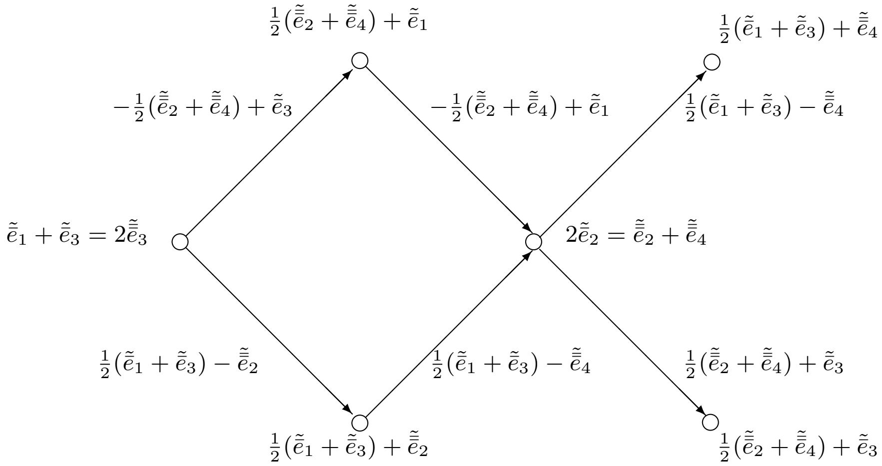


The proof is made up by two steps/identifications. At first
$\bar V=\mathcal{V}(\bar e_1,\bar e_2,\bar e_3)=\mathcal{V}(e_1,e_2,e_3,e_4)/\mathcal{V}(e_2-e_4)$ and
$\bar\alpha_1=\bar e_1-\bar e_2=\bar e_1-\bar e_4=\bar{\tilde{\alpha}}$
$\bar\alpha_2=\bar e_2-\bar e_3=\bar e_4-\bar e_3=-\bar\alpha_3$
$\bar\alpha_3=\bar e_3-\bar e_4=\bar e_3-\bar e_2=-\bar\alpha_2$
$\bar{\bar V}=\mathcal{V}(\bar{\bar e}_2,\bar{\bar e}_3,\bar{\bar e}_4)=\mathcal{V}(e_1,e_2,e_3,e_4)/\mathcal{V}(e_1-e_3)$ and
$\bar{\bar\alpha}_1=\bar{\bar e}_1-\bar{\bar e}_2=\bar{\bar e}_3-\bar{\bar e}_2=-\bar{\bar\alpha}_2$
$\bar{\bar\alpha}_2=\bar{\bar e}_2-\bar{\bar e}_3=\bar{\bar e}_2-\bar{\bar e}_1=-\bar{\bar\alpha}_1$
$\bar{\bar\alpha}_3=\bar{\bar e}_3-\bar{\bar e}_4=\bar{\bar e}_1-\bar{\bar e}_4=\bar{\bar{\tilde{\alpha}}}$.
So we obtain $(\bar A_3^{\rightarrow},\bar\omega_2)$ and $(\bar{\bar A}_3^{\leftarrow},\bar\omega_2)$ respectively, that we picture as follows:



Now we make other identifications:
$\bar{e}_1 + \bar{e}_3 = \bar{\bar{e}}_3 + \bar{\bar{e}}_3$, $\bar{e}_2 + \bar{e}_2 = \bar{\bar{e}}_2 + \bar{\bar{e}}_4$, so one has:
$\bar{\bar{e}}_3 = \frac{1}{2}(\bar{e}_1 + \bar{e}_3)$, $\bar{e}_2 = \frac{1}{2}(\bar{\bar{e}}_2 + \bar{\bar{e}}_4)$ and thus
$\tilde{V} = (\bar{V} \oplus \bar{\bar{V}})/(\bar{e}_1 + \bar{e}_3 - 2\bar{\bar{e}}_3, 2\bar{e}_2 - \bar{\bar{e}}_2 + \bar{\bar{e}}_4) = \mathcal{V}(\tilde{\bar{e}}_1, \tilde{\bar{e}}_3, \tilde{\bar{\bar{e}}}_2, \tilde{\bar{\bar{e}}}_4)$.
We have by direct calculations:
$\tilde{\bar{\alpha}}_2 = \frac{1}{2}(\tilde{\bar{\bar{e}}}_2 + \tilde{\bar{\bar{e}}}_4) - \tilde{\bar{e}}_3$
$\tilde{\bar{\alpha}}_1 = \tilde{\bar{e}}_1 - \frac{1}{2}(\tilde{\bar{\bar{e}}}_2 + \tilde{\bar{\bar{e}}}_4)$
$\tilde{\bar{\bar{\alpha}}}_3 = \frac{1}{2}(\tilde{\bar{e}}_1 + \tilde{\bar{e}}_3) - \tilde{\bar{\bar{e}}}_4$
$\tilde{\bar{\bar{\alpha}}}_2 = -\frac{1}{2}(\tilde{\bar{e}}_1 + \tilde{\bar{e}}_3) + \tilde{\bar{\bar{e}}}_2$
$2\tilde{\bar{\bar{e}}}_3 = \tilde{\bar{e}}_1 + \tilde{\bar{e}}_3$
$\tilde{\bar{e}}_1 + \tilde{\bar{e}}_2 = \tilde{\bar{e}}_1 + \frac{1}{2}(\tilde{\bar{\bar{e}}}_2 + \tilde{\bar{\bar{e}}}_4)$
$2\tilde{\bar{e}}_2 = \bar{\bar{e}}_2 + \bar{\bar{e}}_4$
$\tilde{\bar{\bar{e}}}_3 + \tilde{\bar{\bar{e}}}_2 = \frac{1}{2}(\tilde{\bar{e}}_1 + \tilde{\bar{e}}_3) + \tilde{\bar{\bar{e}}}_2$
$\tilde{\bar{\bar{e}}}_3 + \tilde{\bar{\bar{e}}}_4 = \frac{1}{2}(\tilde{\bar{e}}_1 + \tilde{\bar{e}}_3) + \tilde{\bar{\bar{e}}}_4$
$\tilde{\bar{e}}_2 + \tilde{\bar{e}}_3 = \frac{1}{2}(\tilde{\bar{\bar{e}}}_2 + \tilde{\bar{\bar{e}}}_4) + \tilde{\bar{e}}_3$.

We picture our work and obtain:



The central diagram is a commutative diagram, in fact on one side we have:

$$\tilde{e}_1 + \tilde{e}_3 - (-\frac{1}{2}(\tilde{\tilde{e}}_2 + \tilde{\tilde{e}}_4) + \tilde{e}_3) = \tilde{e}_1 + \frac{1}{2}(\tilde{\tilde{e}}_2 + \tilde{\tilde{e}}_4)$$

$$\tilde{e}_1 + \frac{1}{2}(\tilde{\tilde{e}}_2 + \tilde{\tilde{e}}_4) - (\tilde{e}_1 - \frac{1}{2}(\tilde{\tilde{e}}_2 + \tilde{\tilde{e}}_4) = \tilde{\tilde{e}}_2 + \tilde{\tilde{e}}_4)$$

and on the other side we have:

$$\begin{aligned} 2\tilde{\tilde{e}}_3 - (\frac{1}{2}(\tilde{e}_1 + \tilde{e}_3) - \tilde{\tilde{e}}_2) \\ &= \frac{1}{2}(\tilde{e}_1 + \tilde{e}_3) + \tilde{\tilde{e}}_2 \frac{1}{2}(\tilde{e}_1 + \tilde{e}_3) + \bar{\bar{e}}_2 - (\frac{1}{2}(\tilde{e}_1 + \tilde{e}_3) - \tilde{\tilde{e}}_4) \\ &= \tilde{\tilde{e}}_2 + \tilde{\tilde{e}}_4). \end{aligned}$$

In our vector space the opposite edges are not equal.□

Conjecture 4.1. *The underlying representation corresponding to the weight graph in Construction 4.1 is the direct sum of the second exterior representation and of its dual.*

Construction 4.2. *There are the following oriented weight graphs:*

$$(A_3^{\rightarrow\leftarrow}, \bar{\omega}_2) = (A_3^{\leftarrow}, \bar{\omega}_2) mod(\alpha_1) \odot (A_3^{\rightarrow}, \bar{\omega}_2) mod(\alpha_3) \qquad (5)$$

$$(A_l^{\rightleftarrows}, \bar{\omega}_2) = (A_{l-1}^{\rightleftarrows}, \bar{\omega}_2) mod\alpha_{l-1} \odot (A_l^{\leftrightarrows}, \bar{\omega}_2) mod\alpha_{l-3} \tag{6}$$

$$(A_3^{\rightarrow\leftarrow}, \bar{\omega}_2) = (A_3^{\leftarrow}, \bar{\omega}_2) mod(\alpha_1) \odot (A_3^{\rightarrow}, \bar{\omega}_2) mod(\alpha_3) \tag{7}$$

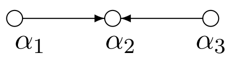
Proof $A_3^{\rightarrow\leftarrow}, \bar{\omega}_2)$ is obtained similarly to Construction 4.1.
To prove (6) we have to turn to a trick, which is an inductive construction. in what follows, we make some choices of orientations, but all of these are without lost of generality. We suppose that
$D_{A_l^{\rightleftarrows}} = D_{A_{l-1}^{\rightleftarrows}} \cup D_{A_2^{\leftarrow}}$, where $\cup$ stands for adds the root α_l, and so on . In such a way we obtain
$D_{A_l^{\rightleftarrows}} = ((D_{A_3^{\rightarrow\leftarrow}} \cup D_{A_2^{\rightarrow}} \cup ...)) \cup D_{A_2^{\leftarrow}}$. Now, from Construction 4.1, we know everything to conclude. □

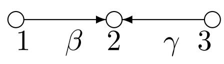
Remarks 4.1. *Our claim is to proof that some finite-dimensional vector spaces generated by weights and roots are the images of certain indecomposable modules (as vector spaces) of ARQ. This case is different from that one in Remark 3.3, since the weights are not the usual ones and the roots coloring the edges are not fundamental. But... the recipe is the same !*

Definition 4.1. *Let L be a semi-simple complex Lie Algebra of type A_3, with Dynkin diagram $A_3^{\leftarrow\rightarrow}$ ($A_3^{\rightarrow\leftarrow}$). Let $K\Gamma$ be the path algebra of a quiver Γ of type A_3 with the same orientation than D_{A_3}. Then is defined a map η from ARQ to $(A_3^{\rightarrow\leftarrow}, \bar{\omega}_2)$ $((A_3^{\leftarrow\rightarrow}, \bar{\omega}_2))$ as follows. Take the class $[P_1]$ which is at the left wringer of ARQ. We pone $\eta([P_1]) = \mathcal{V}(\bar{\bar{e}}_1 + \bar{\bar{e}}_2)$. Then, looking at the arrows of ARQ, we have $\eta([M]) = \mathcal{V}(\bar{\bar{e}}_1 + \bar{\bar{e}}_2, \alpha_1, ..., \alpha_h)$, $\alpha_1, ..., \alpha_h \in \Phi$, $h = 0, 1, 2$, or a quotient of vector spaces of this type.*

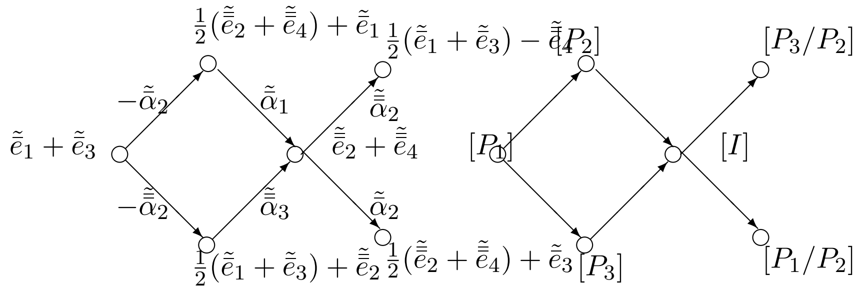
Lemma 4.1. *We refer to definition 4.1. The map η from ARQ to $(A_3^{\rightarrow\leftarrow}, \bar{\omega}_2)$ $((A_3^{\leftarrow\rightarrow}, \bar{\omega}_2))$is an isomorphism, This map is also an isomorphism between oriented edges of $(A_3^{\rightarrow\leftarrow}, \bar{\omega}_2)$ $((A_3^{\leftarrow\rightarrow}, \bar{\omega}_2))$ in an obvious way. Moreover we can select subsystems of roots of type A_2 and sub-quivers of ARQ with relative isomorphisms in the sense above.*

Proof In what follows we consider always the representatives of the classes. The map is well defined. We explicit it.

$D_{A_3^{\rightarrow\leftarrow}}$:


and Γ:


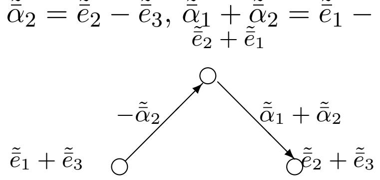
We have respectively the diagram $(A_3^{\rightarrow\leftarrow}, \bar{\omega}_2)$ for $A_3^{\rightarrow\leftarrow}$ and ARQ for $A = KA_3$:



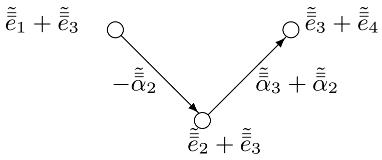
The module P_2 is such that $\dim_K(P_2) = 1$ and $B_K(P_2) = e_2$. Thus $\dim_{\mathbb{C}}(V_{P_2}) = 1$. One has $\eta(P_2) = \mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3)$. ;
The module P_1 is such that $\dim_K(P_1) = 2$ and $B_K(P_1) = \{e_2, \beta\}$. Thus $\dim_{\mathbb{C}}(V_{P_1}) = 2$. One has $\eta(P_1) = \mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3, -\tilde{\tilde{\alpha}}_2)$;
The module P_3 is such that $\dim_K(P_3) = 2$ and $B_K(P_3) = \{e_2, \gamma\}$. Thus $\dim_{\mathbb{C}}(V_{P_3}) = 3$. One has $\eta(P_3) = \mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3, -\tilde{\tilde{\tilde{\alpha}}}_2)$;
The module I is such that $\dim_K(I) = 3$. Thus $\eta(I) = \mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3, -\tilde{\tilde{\alpha}}_2, \tilde{\tilde{\alpha}}_1) \cong \mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3, -\tilde{\tilde{\tilde{\alpha}}}_2, \tilde{\tilde{\tilde{\alpha}}}_3)$;
The module P_3/P_2 is such that $\dim_K(P_3/P_2) = 1$. Thus $\dim_{\mathbb{C}}(V_{P_3/P_2}) = 1$. One has $\eta(P_3/P_2) = \mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3, -\tilde{\tilde{\tilde{\alpha}}}_2)/\mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3)$;
The module P_1/P_2 is such that $\dim_K(P_1/P_2) = 1$. Thus $\dim_{\mathbb{C}}(V_{P_1/P_2}) = 1$. One has $\eta(P_1/P_2) = \mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3, -\tilde{\tilde{\alpha}}_2)/\mathcal{V}(\tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3)$.

The correspondence between maps: in ARQ, the injective morphisms are the arrows with ending points in $[P_1], [P_3], [I]$. These are exactly the proper inclusions with image of codimension 1 in the weight graph. The surjective morphisms in ARQ are the arrows with ending points given by $[P_3/P_2], [P_1/P_2]$; they are exactly the proper quotient maps with kernel of dimension 1. At last, the correspondence between subsystems.

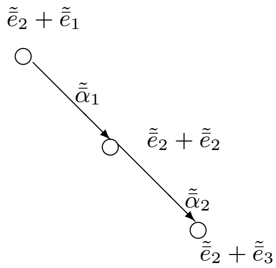
We have the following four blocks :
$\mathcal{B}_1 =< [P_1/P_2], [P_1], [P_2] >$, then there is a vector space $V_{\mathcal{B}_1} = \{\epsilon'_1 = \tilde{\tilde{e}}_1, \epsilon'_2 = \tilde{\tilde{e}}_3, \epsilon'_3\}$, such that $\Pi(A_2) = \{\epsilon'_1 - \epsilon'_2, \epsilon'_2 - \epsilon'_3\}$, $\bar{\omega}_2 = \tilde{\tilde{e}}_1 + \tilde{\tilde{e}}_3$, $\tilde{\tilde{\alpha}}_2 = \tilde{\tilde{e}}_2 - \tilde{\tilde{e}}_3$, $\tilde{\tilde{\alpha}}_1 + \tilde{\tilde{\alpha}}_2 = \tilde{\tilde{e}}_1 - \tilde{\tilde{e}}_3$



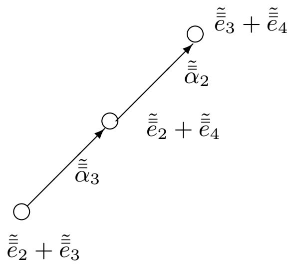
$\mathcal{B}_2 =< [P_1],[P_3/P_2],[P_3] >$, then there is a vector space $V_{\mathcal{B}_2} = \{\epsilon''_1 = \tilde{\tilde{e}}_3, \epsilon''_2 = \tilde{\tilde{e}}_2, \epsilon''_3 = \tilde{\tilde{e}}_4\}$, such that $\Pi(A_2) = \{\epsilon''_1 - \epsilon''_2, \epsilon''_2 - \epsilon''_3\}$, $\bar{\omega}_1 = \tilde{\tilde{e}}_3$, $\tilde{\tilde{\alpha}}_2 = \tilde{\tilde{e}}_2 - \tilde{\tilde{e}}_3$, $\tilde{\tilde{\alpha}}_2 + \tilde{\tilde{\alpha}}_3 = \tilde{\tilde{e}}_2 - \tilde{\tilde{e}}_4$



$\mathcal{B}_3 =< [P_2],[I],[P_1/P_2] >$, then there is a vector space $V_{\mathcal{B}_2} = \{\epsilon^{\cdot}_1 = \tilde{e}_1, \epsilon^{\cdot}_2 = \tilde{e}_2, \epsilon^{\cdot}_3 = \tilde{e}_3\}$, such that $\Pi(A_2) = \{\epsilon^{\cdot}_1 - \epsilon^{\cdot}_2, \epsilon^{\cdot}_2 - \epsilon^{\cdot}_3\}$, $\bar{\omega}_1 = \tilde{e}_1$,$\tilde{\alpha}_1 = \tilde{e}_1 - \tilde{e}_2$, $\tilde{\alpha}_2 = \tilde{e}_2 - \tilde{e}_3$



$\mathcal{B}_4 =< [P_3],[I],[P_2/P_3] >$, then there is a vector space $V_{\mathcal{B}_2} = \{\epsilon^{\cdot\cdot}_1 = \tilde{\tilde{e}}_2, \epsilon^{\cdot\cdot}_2 = \tilde{\tilde{e}}_3, \epsilon^{\cdot\cdot}_3 = \tilde{\tilde{e}}_4\}$, such that $\Pi(A_2) = \{\epsilon^{\cdot\cdot}_1 - \epsilon^{\cdot\cdot}_2, \epsilon^{\cdot\cdot}_2 - \epsilon^{\cdot\cdot}_3\}$, $\bar{\omega}_2 = \tilde{\tilde{e}}_2 + \tilde{\tilde{e}}_3$, $\tilde{\tilde{\alpha}}_3 = \tilde{\tilde{e}}_3 - \tilde{\tilde{e}}_4$, $\tilde{\tilde{\alpha}}_2 = \tilde{\tilde{e}}_2 - \tilde{\tilde{e}}_3$



The dual case is similar$\square$.

Definition 4.2. *Let L be a semi-simple complex Lie Algebra of type A_l, $l \geq 3$, and $D_{A_l^{\rightleftarrows}}$ its non linear oriented Dynkin diagram. Let $K\Gamma$ be the* *path algebra of a quiver* Γ *of type* A_l *with the same orientation than* D_{A_l}. *Then is defined a map* η *from ARQ to* $(A_l^{\overleftarrow{\rightarrow}}, \bar{\omega}_2)$ *as follows. Take the class* $[M]$ *which is at the left wringer (eventually at the bottom on the left) of ARQ. We pone* $\eta([M]) = \mathcal{V}(\bar{e}_1 + \bar{\bar{e}}_2)$. *Then, looking at the arrows of ARQ, we have* $\eta([M]) = \mathcal{V}(\bar{e}_1 + \bar{\bar{e}}_2, \alpha_1, ..., \alpha_h)$, $\alpha_1, ..., \alpha_h \in \Phi$, $h = 0, 1, ..., l-1$, *or a quotient of vector spaces of this type.*

Theorem 4.1. *We refer to Definition 4.2. The map* η *from ARQ to* $(A_l^{\overleftarrow{\rightarrow}}, \bar{\omega}_2)$ *is an isomorphism, also an between oriented edges in an obvious way. Moreover we can select subsystems of roots of type* A_k *and sub-quivers of ARQ with relative isomorphisms in the sense above, for* $2 \leq k \leq l-1$.

*Proof*For $l = 3$ Lemma 4.1holds. Then the result follows by induction on l. □

Remarks 4.2. *We consider (Cf. Section 1.4) the following categories:* $\mathcal{C}_{ARQ,A_l}([M], f)$ *and* $\mathcal{C}_{(A_l^{\overleftarrow{\rightarrow}}, \bar{\omega}_2)}(\mathcal{V}(?), \beta)$, *where* $\mathcal{V}(?)$ *stands for a finite dimensional vector space generated by roots and weights, and* β *stands for an orientated edge as in Construction 4.2.*
Then the map η *in Theorem 4.1 is exactly the functor*
$F : \mathcal{C}_{ARQ,A_l}([M], f) \rightarrow \mathcal{C}_{(A_l^{\overleftarrow{\rightarrow}}, \bar{\omega}_2)}(\mathcal{V}(?), l^?)$ *such that*
$F([M]) = \mathcal{V}(?)$ *with* $dim_{\mathbb{C}}(\mathcal{V}(?)) = dim_{\mathbb{C}}(V_M)$. *Moreover,* $F([M_1] \xrightarrow{\varphi} [M_2]) = F([M_1]) \xrightarrow{l^+} F([M_2])$ *if* φ *is an injective morphism and*
and $F([M_1] \xrightarrow{\varphi} [M_2]) = F([M_1]) \xrightarrow{l^-} F([M_2])$ *if* φ *is a surjective morphism.*
We have $\mathcal{C}_{ARQ,A_l}([M], f) \sim \mathcal{C}_{(A_l^{\leftarrow}, \bar{\omega}_2)}(\mathcal{V}(?), l^?)$ *and*
$\mathcal{C}_{ARQ,A_k}([M], f) \sim \mathcal{C}_{(A_k^{\leftarrow}, \bar{\omega}_2)}(\mathcal{V}(?), l^?)$, *for* $k = 2, 3, ..., l-1$.

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