São Paulo Journal of Mathematical Sciences **7**, 1 (2013), 1–32

On finite W-algebras for Lie algebras and superalgebras

Elena Poletaeva

Department of Mathematics, University of Texas-Pan American, Edinburg, TX 78539

E-mail address: elenap@utpa.edu

Abstract. The finite W-algebras are certain associative algebras associated to a complex semi-simple or reductive Lie algebra $\mathfrak{g}$ and a nilpotent element e of $\mathfrak{g}$. Due to recent results of I. Losev, A. Premet and others, finite W-algebras play a very important role in description of primitive ideals. In the full generality, the finite W-algebras were introduced by A. Premet. It is a result of B. Kostant that for a regular nilpotent (principal) element e, the finite W-algebra coincides with the center of $U(\mathfrak{g})$. Premet's definition makes sense for a simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ in the case when $\mathfrak{g}_{\bar{0}}$ is reductive, $\mathfrak{g}$ admits an invariant super-symmetric bilinear form, and e is an even nilpotent element. We show that certain results of A. Premet can be generalized for classical Lie superalgebras. We consider the case when e is an even regular nilpotent element. The associated finite W-algebra is called principal. Kostant's result does not hold in this case. This is joint work with V. Serganova.

1. Introduction

A *finite W-algebra* is certain associative algebra associated to a complex semi-simple Lie algebra $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$. It is a generalization of the universal enveloping algebra $U(\mathfrak{g})$.

The finite W-algebras are quantizations of Poisson algebras of functions on the Slodowy (i.e. transversal) slice at e to the adjoint orbit $Ad(G)e$, where $\mathfrak{g} = Lie(G)$ [Pr1]. Due to recent results of I. Losev, A. Premet and others, finite W-algebras play a very important role in description of primitive ideals [L2, L3, Pr2, Pr3].

The key ideas for finite W-algebras appeared in the study of classical and quantum affine W-algebras [B, F1, F2]. In the full generality, the finite W-algebras for semi-simple Lie algebras were introduced by A. Premet [Pr1]. The construction of finite W-algebras is based on the study of Whittaker vectors and Whittaker modules in the famous work of B. Konstant [Ko].

This survey was presented during the 3rd Graduate Research Conference in Algebra and Representation Theory at Kansas State University.

Finite W-algebras for Lie algebras have been extensively studied by I. Losev, V. Ginzburg, W. L. Gan, J. Brundan, S. Goodwin, A. Kleshchev, W. Wang and other mathematicians and physicists [L1, BG, GG, BK1, BK2, W].

In Section 2 we discuss Dynkin and good Z-gradings of $\mathfrak{g}$. In Sections 3-5 we review various equivalent definitions of finite W-algebra for $\mathfrak{g}$. In Section 6 we describe Kazhdan filtration on finite W-algebras. In Sections 7 and 8 we give examples of the principal finite W-algebra for $\mathfrak{g} = \mathfrak{gl}(2)$ and $\mathfrak{gl}(n)$.

In Sections 9-11 we recall the definition of a Lie superalgebra, main examples and classification of simple finite-dimensional Lie superalgebras. In Section 12 we review the definition of finite W-algebras for Lie superalgebras. In Section 13 we recall the notion of defect of a basic Lie superalgebra. In section 14 we describe the principal finite W-algebras for Lie superalgebras of type I and defect one. In Section 15 we outline the case when $\mathfrak{g} = \mathfrak{gl}(n|n)$. J.Brown, J.Brundan and S. Goodwin have recently described the principal finite W-algebra of the Lie superalgebra $\mathfrak{gl}(m|n)$ as a certain truncation of a shifted version of the super-Yangian of $\mathfrak{gl}(1|1)$. In Section 16 we review the notion the super-Yangian of $\mathfrak{gl}(m|n)$. In sections 17 and 18 we consider the case when $\mathfrak{g} = Q(n)$. We describe the principal finite W-algebra for $Q(n)$ in terms of generators and relations, and show that it is isomorphic to a factor algebra of the super-Yangian of $Q(1)$. In Section 19 we describe the principal finite W-algebras for the family of simple exceptional Lie superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$. In Sections 20 and 21 we describe the principal finite W-algebra for $\mathfrak{osp}(1|2)$ and obtain partial results and formulate a conjecture for $\mathfrak{osp}(1|2n)$.

2. **Preliminaries**

Let $\mathfrak{g}$ be a finite-dimensional semi-simple or reductive Lie algebra over $\mathbb{C}$ and $(\cdot|\cdot)$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$.

Definition 2.1. *A bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ is $\mathfrak{g}$-invariant if*

$$([x,y] \mid z) = (x \mid [y,z]) \quad \text{for all } x,y,z \in \mathfrak{g}.$$

Definition 2.2. *Adjoint representation of $\mathfrak{g}$.*

For any $x \in \mathfrak{g}$, $ad(x)$ is the endomorphism of $\mathfrak{g}$ defined as follows:

$$ad(x)(y) = [x,y] \text{ for all } y \in \mathfrak{g}.$$

Definition 2.3. *An element $e \in \mathfrak{g}$ is called nilpotent if $ad(e)$ is a nilpotent endomorphism of $\mathfrak{g}$.*

Example 2.1. $\mathfrak{g} = \mathfrak{gl}(n)$, $(a|b) = tr(ab)$.

$e \in \mathfrak{gl}(n)$ is nilpotent if and only if e is an $n \times n$-matrix with eigenvalues zero.

Definition 2.4. *A nilpotent element $e \in \mathfrak{g}$ is regular nilpotent if $\mathfrak{g}^e := Ker(ad(e))$ attains the minimal dimension, which is equal to the rank of $\mathfrak{g}$.*

Example 2.2. $\mathfrak{g} = \mathfrak{gl}(n)$.

$$e = J_n = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{is a regular nilpotent element}.$$

$$\mathfrak{g}^e = < e, e^2, \ldots, e^{n-1}, z >, \quad z = < 1_n >, \quad \dim \mathfrak{g}^e = n.$$

Theorem 2.1. *Jacobson-Morozov [C].*
Associated to a nonzero nilpotent element $e \in \mathfrak{g}$, there always exists an $\mathfrak{sl}(2)$-triple $\{e, h, f\}$ which satisfies

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Proof. Induction on $\dim \mathfrak{g}$.

Example 2.3. $\mathfrak{g} = \mathfrak{gl}(n)$, *$e = J_n$. Let $h = diag(n-1, n-3, \ldots, 3-n, 1-n)$ and*

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{n-1} & 0 \end{pmatrix}, \textit{ where } a_i = i(n-i) \textit{ for } 1 \leq i \leq n-1.$$

Then e, h and f span an $\mathfrak{sl}(2)$.

3. $\mathbb{Z}$-gradings

Definition 3.1. *A Lie algebra $\mathfrak{g}$ is $\mathbb{Z}$-graded if*

$$\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \quad [\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}.$$

Definition 3.2. *A Dynkin $\mathbb{Z}$-grading.*
Let $\mathfrak{sl}(2) = < e, h, f >$. The eigenspace decomposition of the adjoint action

$$ad(h) : \mathfrak{g} \longrightarrow \mathfrak{g}$$

provides a $\mathbb{Z}$-grading: $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, where $\mathfrak{g}_j = \{x \in \mathfrak{g} \mid ad(h)(x) = jx\}$.

A Dynkin $\mathbb{Z}$-grading satisfies the following properties:

(1) $e \in \mathfrak{g}_2$,

(2) $ad(e) : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j+2}$ is injective for $j \leq -1$,

(3) $ad(e) : \mathfrak{g}_j \longrightarrow \mathfrak{g}_{j+2}$ is surjective for $j \geq -1$,

(4) $\mathfrak{g}^e \subset \oplus_{j \geq 0} \mathfrak{g}_j$,

(5) $(\mathfrak{g}_i | \mathfrak{g}_j) = 0$ unless $i + j = 0$,

(6) $\dim \mathfrak{g}^e = \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$.

Proof. (1) follows from the definition of $s\mathfrak{l}(2)$, (2) and (3) follow from the theory of finite-dimensional irreducible $\mathfrak{sl}(2)$-modules, (4), (5) and (6) are easy to prove. We will show that these properties are valid for more general type of $\mathbb{Z}$-gradings, called good $\mathbb{Z}$-gradings.

Definition 3.3. *A good $\mathbb{Z}$-grading.*

A $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ for a semi-simple Lie algebra $\mathfrak{g}$ is good for e, if it satisfies the conditions (1)-(3).

For a reductive $\mathfrak{g}$, there is an additional condition: the center of $\mathfrak{g}$ is in $\mathfrak{g}_0$. Note that good $\mathbb{Z}$-gradings of simple finite-dimensional Lie algebras over an algebraically closed field of characteristic zero were classified in [EK].

Proposition 3.1. *Properties (4)-(6) remain to be valid for every good $\mathbb{Z}$-grading of $\mathfrak{g}$.*

Proof. (see [W])

(4) $\mathfrak{g}^e$ is a $\mathbb{Z}$-graded Lie subalgebra of $\mathfrak{g}$, and (4) follows from (2).

(5) For any $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ there exists a semi-simple element $h \in [\mathfrak{g}, \mathfrak{g}]$ such that

$$\mathfrak{g}_j = \{x \in \mathfrak{g} \mid [h, x] = jx\}.$$

Let $\partial : \mathfrak{g} \longrightarrow \mathfrak{g}$ be the degree operator:

$$\partial(x) := jx \text{ for } x \in \mathfrak{g}_j,$$

then ∂ is a derivation of the semi-simple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, hence ∂ is an inner derivation of $[\mathfrak{g}, \mathfrak{g}]$, given by $ad(h)$ for some semi-simple element $h \in [\mathfrak{g}, \mathfrak{g}]$. Then $\partial = ad(h)$ as derivations of $[\mathfrak{g}, \mathfrak{g}] \oplus \text{center}(\mathfrak{g})$, since the equality hold on the center($\mathfrak{g}$) too. For $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, we have

$$-i(x|y) = ([x, h]|y) = (x|[h, y]) = j(x|y).$$

Then (5) holds.

(6) From (2) and (3),

$$ad(e) : \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_1 \text{ is a bijection.}$$

An exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{g}^e \longrightarrow \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus (\oplus_{j>0}\mathfrak{g}_j) \stackrel{ad(e)}{\longrightarrow} \oplus_{j>0}\mathfrak{g}_j \longrightarrow 0$$

is well-defined by (2) and (3). Then (6) holds.

□

Example 3.1. *Associated to $e = 0$, we have a good $\mathbb{Z}$-grading with $\mathfrak{g}_0 = \mathfrak{g}$.*

Exercise 1. Properties (2) and (3) are equivalent for any $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$.

Remark 3.1. *Obviously, if a $\mathbb{Z}$-grading is Dynkin, then it is a good $\mathbb{Z}$-grading. However, not every good $\mathbb{Z}$-grading is Dynkin.*

Example 3.2. $\mathfrak{g} = \mathfrak{gl}(3)$

$$e = E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad f = E_{31} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that e is a non-regular nilpotent element, $\mathfrak{sl}(2) =< e, h, f >$ defines a Dynkin $\mathbb{Z}$-grading of $\mathfrak{gl}(3)$, whose degrees on the elementary matrices E_{ij} are

$$\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix}.$$

$$\dim(\mathfrak{g}^e) = \dim\mathfrak{g}_0 + \dim\mathfrak{g}_1 = 5, \quad \mathfrak{g}^e =< E_{11}+E_{33}, E_{22}, E_{12}, E_{23}, E_{13} > .$$

Example 3.3. $\mathfrak{g} = \mathfrak{gl}(3)$.

$$e = E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

h defines a good but non-Dynkin $\mathbb{Z}$-grading for e, whose degrees on the elementary matrices E_{ij} are

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}.$$

4. Definition of finite W-algebras

Let $\mathfrak{g}$ be a reductive Lie algebra, $(\cdot|\cdot)$ be a non-degenerate invariant symmetric bilinear form, e be a nilpotent element. Let $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ be a good $\mathbb{Z}$-grading for e. Let $\chi \in \mathfrak{g}^*$ by defined as follows: $\chi(x) := (x|e)$ for $x \in \mathfrak{g}$.

Define a bilinear form on $\mathfrak{g}_{-1}$ by

$$(x,y) := ([x,y] \mid e) = \chi([x,y]) \text{ for } x,y \in \mathfrak{g}_{-1}.$$

Remark 4.1. *The bilinear form on $\mathfrak{g}_{-1}$ is skew-symmetric and non-degenerate.*

Proof. The skew-symmetry follows by definition.
The non-degeneracy follows from the bijection

$$ad(e) : \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_1$$

and the identity

$$(x,y) = ([x,y] \mid e) = (x \mid [y,e]).$$

□

Hence $\dim \mathfrak{g}_{-1}$ is even. Pick a Lagrangian (i.e. a maximal isotropic) subspace $\mathfrak{l}$ of $\mathfrak{g}_{-1}$ with respect to the form $(\cdot,\cdot)$. Then $\dim \mathfrak{l} = \frac{1}{2} \dim \mathfrak{g}_{-1}$.

Let $\mathfrak{m} = (\oplus_{j \leq -2} \mathfrak{g}_j) \oplus \mathfrak{l}$. Note that $\mathfrak{m}$ is a nilpotent subalgebra of $\mathfrak{g}$. The restriction of χ to $\mathfrak{m}$

$$\chi : \mathfrak{m} \longrightarrow \mathbb{C}$$

defines a one-dimensional representation $\mathbb{C}_\chi =< v >$ of $\mathfrak{m}$. In fact, if $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$ and $i \leq -2$ or $j \leq -2$, then $\chi([x,y]) = 0$ by Proposition 1. If $x,y \in \mathfrak{l} \subset \mathfrak{g}_{-1}$, we have that $\chi([x,y]) = (x,y) = 0$ thanks to the Lagrangian condition on $\mathfrak{l}$.

Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $a - \chi(a)$ for all $a \in \mathfrak{m}$.

Definition 4.1. *The generalized Whittaker module is*

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{C}_\chi \cong U(\mathfrak{g})/I_\chi.$$

Definition 4.2. *[Pr1]. The finite W-algebra associated to the nilpotent element e is*

$$W_\chi := End_{U(\mathfrak{g})}(Q_\chi)^{op}.$$

Example 4.1. *Let $e = 0$. Then $\chi = 0$, $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{m} = 0$,*

$$Q_\chi = U(\mathfrak{g}), \quad W_\chi = U(\mathfrak{g}).$$

Theorem 4.1. *B. Kostant (1978) [Ko].*
For a regular nilpotent element $e \in \mathfrak{g}$, $W_\chi \cong Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

Remark 4.2. *The isoclasses of finite W-algebras do not depend on good $\mathbb{Z}$-grading ([BG]) and Lagrangian subspace $\mathfrak{l}$ ([GG]).*

5. The Whittaker model definition of finite W-algebras

We have given endomorphism algebra definition of W_χ. Note that W_χ can also be identified as the space of Whittaker vectors in $U(\mathfrak{g})/I_\chi$.

Definition 5.1. *A $\mathfrak{g}$-module L is called a Whittaker module if $a-\chi(a)$ for all $a \in \mathfrak{m}$ acts on L locally nilpotently. A Whittaker vector in a Whittaker $\mathfrak{g}$-module L is a vector $x \in L$ which satisfies $(a-\chi(a))x = 0$ for all $a \in \mathfrak{m}$.*

Proposition 5.1. *(see [W]). Let $\pi : U(\mathfrak{g}) \to U(\mathfrak{g})/I_\chi$ be the natural projection, and let $y \in U(\mathfrak{g})$. Then*

$$W_\chi = \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid (a-\chi(a))y \in I_\chi \text{ for all } a \in \mathfrak{m}\}.$$

Proof. Since $Q_\chi = U(\mathfrak{g})/I_\chi$ is a cyclic module, then any endomorphism of the $\mathfrak{g}$-module Q_χ is determined by the image of v. The image of v must be annihilated by I_χ. Hence W_χ can be identified as the space of Whittaker vectors in $U(\mathfrak{g})/I_\chi$.

□

By definition of I_χ, W_χ can be further identified with the subspace of $ad\,\mathfrak{m}$-invariants in Q_χ:

$$W_\chi = (Q_\chi)^{ad\mathfrak{m}} := \{\pi(y) \in U(\mathfrak{g})/I_\chi \mid [a,y] \in I_\chi \text{ for all } a \in \mathfrak{m}\}. \tag{1}$$

The algebra structure on W_χ is given by

$$\pi(y_1)\pi(y_2) = \pi(y_1 y_2)$$

for $y_i \in U(\mathfrak{g})$ such that $[a,y_i] \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1,2$.

Exercise 2. Check directly that

(1) The ideal I_χ is $ad\,\mathfrak{m}$-invariant, hence $(Q_\chi)^{ad\mathfrak{m}}$ as a vector space is well-defined.

(2) For y_i satisfying $[a,y_i] \in I_\chi$ for all $a \in \mathfrak{m}$ and $i = 1,2$, we have $[a,y_1y_2] \in I_\chi$ for all $a \in \mathfrak{m}$. Hence $(Q_\chi)^{ad\mathfrak{m}}$ as an algebra is well-defined.

6. Finite W-algebras for even good $\mathbb{Z}$-gradings

Definition 6.1. *A good $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathfrak{g}}\mathfrak{g}_j$ is called even, if $\mathfrak{g}_j = 0$ unless j is an even integer.*

The definition of W_χ for an even good $\mathbb{Z}$-grading is simpler, since in this case $\mathfrak{g}_{-1} = 0$. Hence there is no complications of choice of a Lagrangian subspace $\mathfrak{l}$ and $\mathfrak{m} = \oplus_{j\le -2}\mathfrak{g}_j$.

Let $\mathfrak{p} := \oplus_{j\geq 0}\mathfrak{g}_j$ be a parabolic subalgebra of $\mathfrak{g}$. From the PBW theorem,

$$U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_\chi.$$

The projection

$$pr_\chi : U(\mathfrak{g}) \longrightarrow U(\mathfrak{p})$$

along this direct sum decomposition induces an isomorphism:

$$\overline{pr}_\chi : U(\mathfrak{g})/I_\chi \xrightarrow{\sim} U(\mathfrak{p}).$$

The algebra W_χ can be regarded as a *subalgebra* of $U(\mathfrak{p})$.

Consider a χ-twisted adjoint action of $\mathfrak{m}$ on $U(\mathfrak{p})$ by

$$a \cdot y := pr_\chi([a,y]) \text{ for } a \in \mathfrak{m} \text{ and } y \in U(\mathfrak{p}).$$

Identify W_χ as

$$W_\chi = U(\mathfrak{p})^{ad\mathfrak{m}} := \{y \in U(\mathfrak{p}) \mid [a,y] \in I_\chi \text{ for all } a \in \mathfrak{m}\}.$$

Since $\overline{pr}_\chi : U(\mathfrak{g})/I_\chi \xrightarrow{\sim} U(\mathfrak{p})$ is an isomorphism of $\mathfrak{m}$-modules, this definition is equivalent to the definition (1) (for even $\mathbb{Z}$-gradings) [W].

7. Filtration on W_χ

To introduce a filtration on W_χ, first we will recall the definition of a filtered algebra [A].

Definition 7.1. *A filtered algebra is an algebra A, which has an increasing sequence of subspaces such that*

$$\{0\} \subset F_0 \subset F_1 \subset \ldots \subset F_i \subset \ldots A, \quad A = \bigcup_i F_i,$$

which is compatible with the multiplication:

$$F_i \cdot F_j \subset F_{i+j} \textit{ for all } i, j.$$

As a vector space the associated graded algebra is

$$Gr(A) := \bigoplus_i G_i,$$

where

$$G_i = F_i/F_{i-1} \textit{ for alli} > 0, \quad G_0 = F_0,$$

with multiplication

$$(x + F_{i-1})(y + F_{j-1}) = xy + F_{i+j-1}, \quad x \in F_i, \quad y \in F_j.$$

Definition 7.2. *Kazhdan filtration on W_χ.*
Let $\mathfrak{g}$ be a reductive Lie algebra with a Dynkin $\mathbb{Z}$-grading: $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$, and let $\mathfrak{m} = (\oplus_{j\leq -2}\mathfrak{g}_j)\oplus\mathfrak{l}$. Let $\mathfrak{n}\subset\mathfrak{g}$ be an $ad(h)$-invariant subspace such that $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{n}$ and $\mathbb{C}_\chi = < v >$ be one-dimensional representation of $\mathfrak{m}$. Then

$$W_\chi = \{X \in U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m} \cong S(\mathfrak{n}) \mid aXv = \chi(a)Xv \text{ for all } a \in \mathfrak{m}\}.$$

For any $y \in \mathfrak{n}$, let $wt(y)$ be the weight of y with respect to $ad(h)$ and

$$deg(y) = wt(y) + 2$$

The degree function deg induces a $\mathbb{Z}$-grading on $S(\mathfrak{n})$. This grading defines a filtration on W_χ.

Theorem 7.1. *A. Premet [Pr1].*
The associated graded algebra $Gr(W_\chi)$ is isomorphic to $S(\mathfrak{g}^e)$.

Idea of Proof. Introduce the map

$$P : W_\chi \longrightarrow S(\mathfrak{g}^e).$$

For $X \in W_\chi \subset S(\mathfrak{n})$, let $P(X)$ be the highest weight component in the highest degree component of X, then $P(X)$ belongs to $S(\mathfrak{g}^e)$.

Example 7.1. *If $e = 0$, then the statement of Theorem 3 is the PBW theorem. In this case $W_\chi = U(\mathfrak{g})$, $\mathfrak{g}^e = \mathfrak{g}$ and we have that*

$$Gr(U(\mathfrak{g})) \cong S(\mathfrak{g}).$$

The finite W-algebras associated to *regular* nilpotent elements are called the *principal* finite W-algebras.

8. The case of $\mathfrak{g} = \mathfrak{gl}(2)$

In this section, we describe the principal finite W-algebra for $\mathfrak{gl}(2)$.

Form: $(a|b) = \mathrm{tr}(ab)$.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that e is a regular nilpotent element, h defines an even Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices E_{ij} are

$$\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Let $z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then z is a central element of $\mathfrak{g}$.

$$\mathfrak{g}^e = < z, e >, \quad \dim\mathfrak{g}^e = \dim\mathfrak{g}_0 = 2.$$

$$\mathfrak{m} = \mathfrak{g}_{-2} =< f >, \quad \chi(f) = (f|e) = 1,$$
$$\mathfrak{n} = \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_2, \quad \mathbb{C}_\chi =< v > .$$

W_χ is the polynomial algebra generated by $\pi(z)$ and $\pi(e + \frac{1}{4}h^2 - \frac{1}{2}h)$.

$$\Omega = ef + fe + \frac{1}{2}h^2 \in Z(\mathfrak{g}) \text{ is the quadratic Cazimir element of } \mathfrak{g}.$$

We have that

$$\frac{1}{2}\pi(\Omega) = \frac{1}{2}\pi(ef + fe + \frac{1}{2}h^2)) = \frac{1}{2}\pi(2ef - h + \frac{1}{2}h^2)) = \pi(e + \frac{1}{4}h^2 - \frac{1}{2}h).$$

The generators of W_χ can be identified with elements of $\mathfrak{g}^e$:

$$\pi(z) \xrightarrow{P} z,$$
$$\frac{1}{2}\pi(\Omega) \xrightarrow{P} e.$$

9. The case of $\mathfrak{g} = \mathfrak{gl}(n)$

In this section, we describe the principal finite W-algebra for $\mathfrak{gl}(n)$.

Form: $(a|b) = \mathrm{tr}(ab)$. Let

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ n-1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2(n-2) & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-1 & 0 \end{pmatrix},$$

$h = \mathrm{diag}(n-1, n-3, \ldots, 3-n, 1-n)$. Note that e is a regular nilpotent element, and h defines an even Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices E_{ij} are

$$\begin{pmatrix} 0 & 2 & 4 & 6 & \cdots & 2n-2 \\ -2 & 0 & 2 & 4 & \cdots & 2n-4 \\ -4 & -2 & 0 & 2 & \cdots & 2n-6 \\ -6 & -4 & -2 & 0 & \cdots & 2n-8 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & -6 & -4 & -2 & 0 \end{pmatrix}.$$

Let $z = \mathrm{diag}(1, \ldots, 1)$ be a central element of $\mathfrak{gl}(n)$. Then

$$\mathfrak{g}^e =< z, e, e^2, e^3, \ldots, e^{n-1} >, \quad \dim \mathfrak{g}^e = \dim \mathfrak{g}_0 = n.$$

$$\mathfrak{m} = \bigoplus_{j \geq 2}^{n} \mathfrak{g}_{2-2j}, \quad \chi(E_{i+1,i}) = 1, \quad \chi(E_{i+k,i}) = 0 \text{ if } k \geq 2.$$

Recall that the *Casimir elements* of $\mathfrak{g}$ are generators of $Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$. If $\mathfrak{g} = \mathfrak{gl}(n)$ they are given as follows:

$$\Omega_k = \sum_{i_1,i_2,\ldots,i_k} E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_k i_1}, \text{ for } k \geq 1.$$

Note that W_χ is the polynomial algebra generated by n elements:

$$\pi(z), \pi(\Omega_2), \pi(\Omega_3), \ldots, \pi(\Omega_n).$$

The generators of W_χ can be identified with elements of $\mathfrak{g}^e$:

$$\pi(z) \xrightarrow{P} z,$$
$$\tfrac{1}{k}\pi(\Omega_k) \xrightarrow{P} e^{k-1} \text{ for } k = 2, \ldots, n.$$

10. Lie superalgebras

Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$.

Definition 10.1. *A superspace is a $\mathbb{Z}_2$-graded vector space*

$$V = V_{\bar{0}} \oplus V_{\bar{1}}.$$

Definition 10.2.
The dimension of a superspace: $\dim V = (m|n)$*, where* $\dim V_{\bar{0}} = m$*,* $\dim V_{\bar{0}} = n$*.*
The parity of an element: if $v \in V_{\bar{0}}$ then $p(v) = \bar{0}$, if $v \in V_{\bar{1}}$ then $p(v) = \bar{1}$.

Definition 10.3. *A superalgebra is a $\mathbb{Z}_2$-graded algebra*

$$A = A_{\bar{0}} \oplus A_{\bar{1}}, \quad A_i A_j \subseteq A_{i+j} \text{ for } i, j \in \mathbb{Z}_2.$$

Definition 10.4. *A Lie superalgebra is a $\mathbb{Z}_2$-graded algebra*

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

with an operation $[\ ,\ ]$ satisfying the following axioms:
1. super-anticommutativity:

$$[x,y] = -(-1)^{p(x)p(y)}[y,x] \quad \text{for all } x,y \in \mathfrak{g},$$

2. super-Jacobi identity:

$$[x,[y,z]] = [[x,y],z] + (-1)^{p(x)p(y)}[y,[x,z]] \quad \text{for all } x,y,z \in \mathfrak{g}.$$

Remark 10.1. *$\mathfrak{g}_{\bar{0}}$ is a Lie algebra, $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{1}}$, hence $\mathfrak{g}_{\bar{1}}$ is a module over $\mathfrak{g}_{\bar{0}}$, $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] \subseteq \mathfrak{g}_{\bar{0}}$.*

Definition 10.5. *A Lie superalgebra $\mathfrak{g}$ is simple, if it is not abelian and the only $\mathbb{Z}_2$-graded ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$.*

Remark 10.2. *Sign Rule. If an element x with parity $p(x)$ moves through an element y with parity $p(y)$, then the sign $(-1)^{p(x)p(y)}$ appears in the formula.*

11. Main examples of Lie superalgebras

Example 11.1. *The General Linear Lie superalgebra.*

$$\mathfrak{g} = \mathfrak{gl}(m|n) = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, \textit{ where}$$

$$\mathfrak{g}_{\bar{0}} = \{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right) \mid A \textit{ is a } m \times m \textit{ matrix, } B \textit{ is a } n \times n \textit{ matrix} \},$$

$$\mathfrak{g}_{\bar{1}} = \{ \left(\begin{array}{c|c} 0 & C \\ \hline D & 0 \end{array}\right) \mid C \textit{ is a } m \times n \textit{ matrix, } D \textit{ is a } n \times m \textit{ matrix} \}.$$

Definition 11.1. *Super-bracket:*

$$[X,Y] = XY - (-1)^{p(X)p(Y)} YX \textit{ for } X,Y \in \mathfrak{g}.$$

Super-trace:

$$str \left(\begin{array}{c|c} A & C \\ \hline D & B \end{array}\right) = tr(A) - tr(B).$$

Example 11.2. *The Special Linear Lie superalgebra:*

$$\mathfrak{sl}(m|n) := \{X \in \mathfrak{gl}(m|n) \mid strX = 0\}.$$

Note that $\mathfrak{sl}(m|n)$is simple if and only if $m \neq n$. If $m = n$ then $\mathfrak{sl}(n|n)/ < 1_{2n} >$ is simple.

Example 11.3. *The Orthogonal-Symplectic Lie superalgebra.*
Let F be a non-degenerate super symmetric bilinear form on a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\dim V = (m|n)$*:*

$$F(v,w) = (-1)^{p(v)p(w)} F(w,v) \qquad v,w \in V.$$

The restriction of F to $V_{\bar{0}}$ is symmetric, and to $V_{\bar{1}}$ is skew-symmetric, hence n is even.

$$\mathfrak{osp}(m|n) := \{X \in \mathfrak{gl}(m|n) \mid F(X(v),w) + (-1)^{p(X)p(v)} F(v,X(w)) = 0, \ \forall v,w \in V\}.$$

For instance, let $m = 2l, \quad n = 2r$.

$$F = \left(\begin{array}{cc|cc} 0 & 1_l & 0 & 0 \\ 1_l & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1_r \\ 0 & 0 & -1_r & 0 \end{array}\right),$$

where 1_l is the identity $l\times l$-matrix, 1_r is the identity $r\times r$-matrix. Then

$$\mathfrak{osp}(m|n)=\left(\begin{array}{cc|cc} A & B & X & P\\ C & -A^t & Y & Q\\ \hline Q^t & P^t & D & E\\ -Y^t & -X^t & F & -D^t\end{array}\right),$$

where A is any $l\times l$ matrix, B and C are skew-symmetric $l\times l$ matrices, D is any $r\times r$ matrix, E and F are symmetric $r\times r$ matrices, X,Y,P,Q are any $l\times r$ matrices.

12. Classification of simple finite-dimensional Lie superalgebras

Definition 12.1. *[K]. A Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ is called classical, if it is simple, and the representation of the Lie algebra $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}_{\bar 1}$ is completely reducible.*

Definition 12.2. *A classical Lie superalgebra $\mathfrak{g}$ is called basic, if $\mathfrak{g}$ admits an even non-degenerate $\mathfrak{g}$-invariant bilinear form.*

Definition 12.3. *A classical Lie superalgebra $\mathfrak{g}$ is of Type I, if $\mathfrak{g}_{\bar 1}$ is a direct sum if two simple $\mathfrak{g}_{\bar 0}$-submodules.*

Remark 12.1. *Notations:*

$A(m,n)=\mathfrak{sl}(m+1|n+1)$ *for* $m\neq n,\quad m,n\geq 0,$

$A(n,n)=\mathfrak{sl}(n+1|n+1)/<1_{2n+2}>,\quad n>0,$

$B(m,n)=\mathfrak{osp}(2m+1|2n),\quad m\geq 0,n>0,$

$D(m,n)=\mathfrak{osp}(2m|2n),\quad m\geq 2,n>0,$

$C(n)=\mathfrak{osp}(2|2n-2),\quad n\geq 2.$

Theorem 12.1. *V. G. Kac (1977) [K].*

A simple finite-dimensional Lie superalgebra over an algebraically closed field of characteristic zero is isomorphic either to one of the simple Lie algebras, or to one of classical Lie superalgebras:

$$A(m,n),B(m,n),C(n),D(m,n),D(2,1;\alpha),F(4),G(3),P(n),\tilde{Q}(n),$$

or to one of Lie superalgebras of Cartan type: $W(n),S(n),H(n),\tilde{S}(n)$.

13. Finite W-algebras for Lie superalgebras (joint work with V. Serganova)

Finite W-algebras for Lie superalgebras have been studied by C. Briot, E. Ragoucy, J. Brundan, J. Brown, S. Goodwin, W. Wang, L. Zhao and other mathematicians and physicists [BR, BBG, W, Z]. Analogues of finite W-algebras for Lie superalgebras in terms of BRST cohomology were defined in [DK].

We consider the case when $\mathfrak{g}$ is a classical simple Lie superalgebra, i.e. $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, and $\mathfrak{g}$ has an invariant super-symmetric bilinear form $(\cdot|\cdot)$.

Remark 13.1. *The Premet's definition makes sense for Lie superalgebras. One should consider a good $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_j$ for $e \in (\mathfrak{g}_2)_{\bar{0}}$ (i.e. a $\mathbb{Z}$-grading which satisfies properties (2) and (3) of Def. 6). Note that good $\mathbb{Z}$-gradings of basic Lie superalgebras over an algebraically closed field of characteristic zero were classified in [H].*

Let $e \in \mathfrak{g}_{\bar{0}}$ be an even nilpotent element, and we fix $\mathfrak{sl}(2) =< e,h,f >$. As in the Lie algebra case, the linear operator $ad(h)$ defines a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{j\in\mathbb{Z}}\mathfrak{g}_i$. Let $\mathfrak{g}^e := \mathrm{Ker}(ad(e))$. Note that $\dim\mathfrak{g}^e = \dim\mathfrak{g}_0 + \dim\mathfrak{g}_1$. Let $\mathfrak{l}$ be a Lagrangian subspace in $\mathfrak{g}_{-1}$, with respect to the super-skew-symmetric bilinear form $(x,y) = ([x,y]|e)$. We consider a nilpotent subalgebra $\mathfrak{m} = \oplus_{j\leq -2}\mathfrak{g}_j \oplus \mathfrak{l}$ of $\mathfrak{g}$. Let $\chi \in \mathfrak{g}^*$ be defined by $\chi(x) = (x|e)$. Let $\mathbb{C}_\chi =< v >$ be the one-dimensional $\mathfrak{m}$-module with character χ. Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $a-\chi(a)$ for all $a \in \mathfrak{m}$.

The *generalized Whittaker module* Q_χ is defined as in Def.8. The *finite W-algebra* W_χ associated to the nilpotent element e is defined as in Def.9.

Remark 13.2.

(1) The isoclasses of W_χ do not depend on the choice of Lagrangian subspace $\mathfrak{l}$ [Z].

(2) Theorem of Kostant does not hold for Lie superalgebras, since W_χ must have a non-trivial odd part, and the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ is even.

(3) Kazhdan filtration on W_χ can be defined exactly as in the Lie algebra case (see Def. 13).

Proposition 13.1. *$Gr(W_\chi)$ is supercommutative.*

Remark 13.3. *If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, then one can construct the similar map*

$$P : W_\chi \longrightarrow S(\mathfrak{g}^e)$$

by taking the highest weight component in the highest degree component.

If $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is odd, then there exists an odd element θ in $\mathfrak{g}_{-1} \cap \mathfrak{l}^\perp$ such that $\pi(\theta) \in W_\chi$ and $\pi(\theta)^2 = 1$.

In what follows we study the *principal* finite W-algebras, which are the finite W-algebras associated to even *regular* nilpotent elements.

14. Defect

Definition 14.1. *[KW].*

Let $\mathfrak{g}$ be a classical Lie superalgebra, and let $\triangle$ be the set of roots with respect to a maximal torus in $\mathfrak{g}_{\bar{0}}$. If $\mathfrak{g}$ is a basic Lie superalgebra, then the defect of $\mathfrak{g}$ def$(\mathfrak{g})$ is the dimension of a maximal isotropic subspace in the $\mathbb{R}$-span of $\triangle$.

Example 14.1.

$$def(\mathfrak{sl}(m|n)) = min(m,n),$$
$$def(\mathfrak{osp}(2m|2n)) = def(\mathfrak{osp}(2m+1|2n)) = min(m,n).$$

The exceptional Lie superalgebras

$$D(2,1;\alpha), G(3), F(4)$$

have defect one.

Remark 14.1. *[PS1]. If e is a regular nilpotent element in $\mathfrak{g}$, then*

$$\dim(\mathfrak{g}^e)_{\bar{1}} = 2def(\mathfrak{g}) \text{ or } 2def(\mathfrak{g})+1.$$

$\dim(\mathfrak{g}^e)_{\bar{1}} = 2def(\mathfrak{g})$, *if* $\mathfrak{g} = \mathfrak{sl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$, $m \geq n$; $\mathfrak{osp}(2m|2n)$, $m \leq n$, $G(3)$;

$\dim(\mathfrak{g}^e)_{\bar{1}} = 2def(\mathfrak{g})+1$, *if* $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$, $m < n$; $\mathfrak{osp}(2m|2n)$, $m > n$; $D(2,1;\alpha)$, $F(4)$.

15. **The case of $\mathfrak{g} = \mathfrak{sl}(1|n)$**

In this section, we describe the principal finite W-algebra for $\mathfrak{sl}(1|n)$ and more generally, for a Lie superalgebra of Type I and defect one.

Form: $(a|b) = -\mathrm{str}(ab)$

$$e = \left(\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \quad f = \left(\begin{array}{c|ccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & n-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2(n-2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n-1 & 0 \end{array}\right)$$

$$h = \mathrm{diag}(0|n-1, n-3, \ldots, 3-n, 1-n).$$

Note that e is a regular nilpotent element. We will use the following notations for some elementary matrices in $\mathfrak{gl}(1|n)$:

$$E_{ij} = \left(\begin{array}{c|ccccc} h_0 & \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\ \hline \xi_1 & h_1 & e_1 & \cdots & \cdots & \cdots \\ \xi_2 & f_1 & h_2 & e_2 & \cdots & \cdots \\ \xi_3 & \cdots & f_2 & h_3 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & e_{n-1} \\ \xi_n & \cdots & \cdots & \cdots & f_{n-1} & h_n \end{array}\right).$$

h defines a Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices E_{ij} are

$$\left(\begin{array}{c|cccccc} 0 & 1-n & 3-n & \cdots & n-3 & \mathbf{n-1} \\ \hline \mathbf{n-1} & 0 & 2 & 4 & \cdots & 2n-2 \\ n-3 & -2 & 0 & 4 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 3-n & 4-2n & \cdots & \cdots & 0 & 2 \\ 1-n & 2-2n\cdots & \cdots & \cdots & -2 & 0 \end{array}\right).$$

Let $c = \mathrm{diag}(n|1,\cdots,1)$ be a central elements of $\mathfrak{g}_{\bar{0}}$. Then

$$\mathfrak{g}^e =< e, e^2, \cdots, e^{n-1}, c \mid \xi_1, \mu_n >, \quad \dim(\mathfrak{g}^e) = (n|2).$$

Let $\mathfrak{m} = (\bigoplus_{j\leq -2}\mathfrak{g}_j)\bigoplus\mathfrak{l}$. Note that if n is odd, then $\mathfrak{l} = 0$, and if $n = 2k$, then $\mathfrak{l} =< \xi_{k+1} >$. Note also that $\mathfrak{m}$ is generated by $f_1,\ldots,f_n;\mu_1,\ldots,\mu_{k-1},\xi_{k+1},\ldots,\xi_n$, if $n = 2k$, and by $f_1,\ldots,f_n;\mu_1,\ldots,\mu_k,\xi_{k+2},\ldots,\xi_n$, if $n = 2k+1$.

$$\chi(f_i) = 1 \quad \chi(\mu_i) = \chi(\xi_i) = 0.$$

Let $\mathfrak{n}$ be an $ad(h)$-invariant subspace of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{m}\bigoplus\mathfrak{n}$. Recall that $\mathfrak{g}$ admits $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}^{-1}\oplus\mathfrak{g}^0\oplus\mathfrak{g}^1$ consistent with the $\mathbb{Z}_2$-grading:

$$\mathfrak{g}^0 = \left(\begin{array}{c|ccc} * & 0 & \ldots & 0 \\ \hline 0 & * & * & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & * & * & * \end{array}\right), \quad \mathfrak{g}^1 = \left(\begin{array}{c|ccc} 0 & * & \cdots & * \\ \hline 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$\mathfrak{g}^{-1} = \left(\begin{array}{c|ccc} 0 & 0 & \cdots & 0 \\ \hline * & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ * & 0 & 0 & 0 \end{array}\right)$$

Fix an $ad(h)$-homogeneous bases:

$B(\mathfrak{m}_{-1})$ of $\mathfrak{m}\cap\mathfrak{g}^{-1}$, $B(\mathfrak{m}_1)$ of $\mathfrak{m}\cap\mathfrak{g}^1$, $B(\mathfrak{n}_{-1})$ of $\mathfrak{n}\cap\mathfrak{g}^{-1}$, $B(\mathfrak{n}_1)$ of $\mathfrak{n}\cap\mathfrak{g}^1$.

Set

$$R_1 = (\prod_{x\in B(\mathfrak{m}_1)} x)(\prod_{y\in B(\mathfrak{n}_{-1})} y),$$
$$R_2 = (\prod_{y\in B(\mathfrak{m}_{-1})} y)(\prod_{x\in B(\mathfrak{n}_1)} x).$$

Note that $\pi(R_1)$ and $\pi(R_2)$ are both Whittaker vectors, hence $\pi(R_1),\pi(R_2)\in W_\chi$.

Let

$$p(E_{ij}) = p(i) + p(j),$$

where E_{ij} is an elementary matrix, and

$$p(i) = \begin{cases} 0 & if \quad 1\leq i\leq m, \\ 1 & if \quad m+1\leq i\leq m+n. \end{cases}$$

Remark 15.1. *Casimir elements of* $\mathfrak{gl}(m|n)$ *are*

$$\Omega_k = \sum_{i_1, i_2, \ldots, i_k} (-1)^{p(i_2)+\cdots+p(i_k)} E_{i_1 i_2} E_{i_2 i_3} \ldots E_{i_k i_1}.$$

Even generators of W_χ are $\pi(\Omega_k)$, for $k = 2, \ldots n$ and $\pi(c)$.
The generators of W_χ can be identified with elements of $\mathfrak{g}^e$:

$$\begin{aligned}
&(-1)^{k+1}\tfrac{1}{k}\pi(\Omega_k) \xrightarrow{P} e^{k-1}, \quad k = 2, \ldots, n,\\
&\pi(c) \xrightarrow{P} c,\\
&\pi(R_1) \xrightarrow{P} \xi_1,\\
&\pi(R_2) \xrightarrow{P} \mu_n.
\end{aligned}$$

Theorem 15.1. *[PS1]. Let* $\mathfrak{g}$ *be a Lie superalgebra of Type I and defect one (i.e.* $\mathfrak{g} = \mathfrak{sl}(1|n)$ *or* $\mathfrak{osp}(2|2n-2)$*). Let* n *be the rank of* $\mathfrak{g}_{\bar{0}}$*,* c *be a central element of* $\mathfrak{g}_{\bar{0}}$*, and* $\Omega_2, \ldots, \Omega_n$ *be the first* $n-1$ *Casimir elements in* $Z(\mathfrak{g})$*. Then the principal finite* W*-algebra* W_χ *is a finite extension of* $\mathbb{C}[\pi(c), \pi(\Omega_2), \ldots, \pi(\Omega_n)]$ *with odd generators* $\pi(R_1), \pi(R_2)$ *and defining relations*

$$\begin{gathered}
\pi(R_1)^2 = \pi(R_2)^2 = 0, \quad [\pi(c), \pi(R_1)] = -\pi(R_1), \quad [\pi(c), \pi(R_2)] = \pi(R_2),\\
[\pi(\Omega_i), \pi(R_1)] = [\pi(\Omega_i), \pi(R_2)] = 0, \quad i = 2, \ldots, n,\\
[\pi(R_1), \pi(R_2)] = \pi(\tilde{\Omega}),
\end{gathered}$$

where

$$\tilde{\Omega} = \prod_{y \in B(\mathfrak{g}^{-1})} ad\, y(\prod_{x \in B(\mathfrak{g}^1)} x)$$

is an element of $Z(\mathfrak{g})$*. In this case* $W_\chi \cong U(\mathfrak{g}^e)$*.*

16. **The case of** $\mathfrak{g} = \mathfrak{gl}(n|n)$

In this section, we outline the principal finite W-algebra for $\mathfrak{gl}(n|n)$.

Form: $(a|b) = \mathrm{str}(ab)$. We will use the following notations for some elementary matrices in $\mathfrak{gl}(n|n)$:

$$\left(\begin{array}{ccccc|ccccc} h_1 & e_1 & \cdots & \cdots & \cdots & y_1 & y_{n+1} & \cdots & \cdots & \cdots \\ f_1 & h_2 & e_2 & \cdots & \cdots & \mu_1 & y_2 & y_{n+2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & f_{n-2} & h_{n-1} & e_{n-1} & \cdots & \cdots & \mu_{n-2} & y_{n-1} & y_{2n-1} \\ \cdots & \cdots & \cdots & f_{n-1} & h_n & \cdots & \cdots & \cdots & \mu_{n-1} & y_n \\ \hline x_1 & x_{n+1} & \cdots & \cdots & \cdots & h_{n+1} & e_n & \cdots & \cdots & \cdots \\ \xi_1 & x_2 & x_{n+2} & \cdots & \cdots & g_1 & h_{n+2} & e_{n+1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \xi_{n-2} & x_{n-1} & x_{2n-1} & \cdots & \cdots & g_{n-2} & h_{2n-1} & e_{2n-2} \\ \cdots & \cdots & \cdots & \xi_{n-1} & x_n & \cdots & \cdots & \cdots & g_{n-1} & h_{2n} \end{array}\right)$$

Let $\mathfrak{sl}(2) =< e,h,f >$, where $e = e_1 + e_2 + \cdots e_{2n-2}, f = \sum_{i=1}^{n-1} i(n-i)(f_i + g_i)$, $h = \mathrm{diag}(n-1, n-3, \cdots, 1-n \mid n-1, n-3, \cdots, 1-n)$. Note that e is a regular nilpotent element. Let $z = \mathrm{diag}(1,\cdots,1|1,\cdots,1)$, $c = \mathrm{diag}(1,\cdots,1|-1,\cdots,-1)$. Note that $\dim(\mathfrak{g}^e) = (2n|2n)$. Explicitly, $\mathfrak{g}^e = \mathfrak{g}^e_{\bar 0} \oplus \mathfrak{g}^e_{\bar 1}$, where

$\mathfrak{g}^e_{\bar 0} =< (e_1 + \cdots + e_{n-1})^i, (e_n + \cdots + e_{2n-2})^i, z, c >,$

$\mathfrak{g}^e_{\bar 1} =< (x_1 + \cdots + x_n), (y_1 + \cdots + y_n), (x_{n+1} + \cdots + x_{2n-1})^i, (y_{n+1} + \cdots + y_{2n-1})^i >,$

where $i = 1, \ldots, n-1$, and the powers are considered in the corresponding $n \times n$ matrices.

Theorem 16.1. *[PS1]. In the case when $\mathfrak{g} = \mathfrak{gl}(n|n)$, $\mathfrak{g}^e$ is isomorphic to the truncated Lie superalgebra of polynomial currents in $\mathfrak{gl}(1|1)$.*

$$\mathfrak{gl}(1|1) = \{ \left(\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{21} & a_{22} \end{array}\right) \mid a_{ij} \in \mathbb{C}\}.$$

The isomorphism

$$\varphi : \mathfrak{g}^e \longrightarrow \mathfrak{gl}(1|1) \otimes \mathbb{C}[t]/(t^n)$$

is given as follows:

$\varphi((e_1 + \cdots + e_{n-1})^i) = E_{11} \otimes t^i,$ $\quad \varphi((e_n + \cdots + e_{2n-2})^i) = E_{22} \otimes t^i,$

$\varphi((x_{n+1} + \cdots + x_{2n-1})^i) = E_{21} \otimes t^i,$ $\quad \varphi((y_{n+1} + \cdots + y_{2n-1})^i) = E_{12} \otimes t^i,$

for $i = 1, \ldots, n-1$,

$$\varphi(\frac{z+c}{2}) = E_{11},\ \varphi(\frac{z-c}{2}) = E_{22},\ \varphi(x_1 + \cdots + x_n) = E_{21},\ \varphi(y_1 + \cdots + y_n) = E_{12}.$$

h defines an even Dynkin $\mathbb{Z}$-grading of $\mathfrak{gl}(n|n)$ whose degrees on the elementary matrices are

$$\left(\begin{array}{cccc|cccc} 0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\ -2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \\ \hline 0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\ -2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \end{array}\right).$$

$$\mathfrak{m} = \bigoplus_{j=2}^{n} \mathfrak{g}_{2-2j},$$

$\mathfrak{m}$ is generated by ξ_i, μ_i, f_i, g_i for $i = 1, \cdots, n-1$ Note that

$$\chi(f_i) = -\chi(g_i) = 1; \qquad \chi(\mu_i) = \chi(\xi_i) = 0, \text{ for } i = 1, \cdots, n-1.$$

Note that W_χ is generated by $2n$ even elements and $2n$ odd elements.

The generators of W_χ can be identified with elements of $\mathfrak{g}^e$ using the map

$$P : W_\chi \longrightarrow S(\mathfrak{g}^e).$$

17. The super-Yangian of $\mathfrak{gl}(m|n)$

In this section, we outline the correspondence between finite W-algebras for $\mathfrak{gl}(m|n)$ and super-Yangians.

Recall that for a finite-dimensional semi-simple Lie algebra $\mathfrak{g}$, the Yangian of $\mathfrak{g}$ is an infinite-dimensional Hopf algebra $Y(\mathfrak{g})$. It is a deformation of the universal enveloping algebra of the Lie algebra of polynomial currents of $\mathfrak{g}$ [M].

Definition 17.1. *The super-Yangian $Y(\mathfrak{gl}(m|n))$ of $\mathfrak{gl}(m|n)$ is an associative unital superalgebra over $\mathbb{C}$ with a countable set of generators*

$$T_{i,j}^{(1)}, T_{i,j}^{(2)}, \ldots, \text{ where } i, j = 1, \ldots, m+n.$$

The $\mathbb{Z}_2$-grading of the algebra $Y(\mathfrak{gl}(m|n))$ is defined as follows:

$$p(T_{i,j}^{(n)}) = p(i) + p(j).$$

To write down defining relation for the generators of $Y(\mathfrak{gl}(m|n))$ we employ the formal series in $Y(\mathfrak{gl}(m|n)[[u^{-1}]]$:

$$T_{i,j}(u) = \delta_{ij} \cdot 1 + T_{i,j}^{(1)} u^{-1} + T_{i,j}^{(2)} u^{-2} + \ldots.$$

For all indices i, j, k, l we have the relations

$$(u-v)[T_{i,j}(u),T_{k,l}(v)] =$$
$$(-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}((T_{k,j}(u)T_{i,l}(v)-T_{k,j}(v)T_{i,l}(u)), \qquad (2)$$

where v is a formal parameter independent of u, so that (2) is an equality in the algebra of formal Laurent series in u^{-1},v^{-1} with coefficients in $Y(\mathfrak{gl}(m|n))$.

The following Proposition follows from [BR].

Proposition 17.1. *In the case when $\mathfrak{g}=\mathfrak{gl}(n|n)$, the corresponding principal finite W-algebra is isomorphic to the truncated super-Yangian $Y(\mathfrak{gl}(1|1))/(n)$.*

Remark 17.1. *The principal finite W-algebras for $\mathfrak{gl}(m|n)$ associated to regular nilpotent elements were described as certain truncations of a shifted version of the super-Yangian $Y(\mathfrak{gl}(1|1))$ by J. Brown, J. Brundan and S. Goodwin in 2012 [BBG]. They also classified irreducible modules over principal finite W-algebras for $\mathfrak{gl}(m|n)$ by highest weight theory and proved that they are finite-dimensional.*

In 2003 C. Briot and E. Ragoucy observed that certain finite W-algebras based on $\mathfrak{gl}(m|n)$ can be realized as truncations of the super-Yangian of $\mathfrak{gl}(m|n)$ [BR]. They also observed that finite W-algebras for $\mathfrak{gl}(m|n)$ associated to non-regular nilpotent elements are connected to higher rank super-Yangians.

18. The case of $\mathfrak{g}=\mathbf{Q(n)}$

In this section, we construct a complete set of generators of the principal finite W-algebra for $Q(n)$.

The *queer* Lie superalgebra is defined as follows

$$Q(n) := \{ \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) \mid A,B \text{ are } n\times n \text{ matrices}\}.$$

Let $\mathrm{otr}\left(\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right) = \mathrm{tr} B$.

Remark 18.1. *$Q(n)$ has one-dimensional center $<z>$, where $z=1_{2n}$. Let*

$$SQ(n)=\{X\in Q(n)\mid otrX=0\}.$$

Note that the Lie superalgebra $\tilde{Q}(n):=SQ(n)/<z>$ is simple.

Let $e_{i,j}$ and $f_{i,j}$ be standard bases in A and B respectively:

$$e_{i,j}=\left(\begin{array}{c|c} E_{ij} & 0 \\ \hline 0 & E_{ij} \end{array}\right), \quad f_{i,j}=\left(\begin{array}{c|c} 0 & E_{ij} \\ \hline E_{ij} & 0 \end{array}\right),$$

where E_{ij} are elementary $n\times n$ matrices.

Note that $\mathfrak{g} = Q(n)$ admits an *odd* non-degenerate $\mathfrak{g}$-invariant super symmetric bilinear form

$$(x|y) := \mathrm{otr}(xy) \text{ for } x, y \in \mathfrak{g}.$$

Therefore, we identify the coadjoint module $\mathfrak{g}^*$ with $\Pi(\mathfrak{g})$, where Π is the functor changing the parity.

Let $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = \sum_{i=1}^{n-1} e_{i,i+1}, \quad h = \mathrm{diag}(n-1, n-3, \dots, 3-n, 1-n), \quad f = \sum_{i=1}^{n-1} i(n-i)e_{i+1,i}.$$

Note that e is a regular nilpotent element, h defines an even Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices are

$$\left(\begin{array}{cccc|cccc} 0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\ -2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \\ \hline 0 & 2 & \cdots & 2n-2 & 0 & 2 & \cdots & 2n-2 \\ -2 & 0 & \cdots & 2n-4 & -2 & 0 & \cdots & 2n-4 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 2-2n & \cdots & \cdots & 0 & 2-2n & \cdots & \cdots & 0 \end{array}\right).$$

Let $E = \sum_{i=1}^{n-1} f_{i,i+1}$. Since we have an isomorphism $\mathfrak{g}^* \cong \Pi(\mathfrak{g})$, an even regular nilpotent $\chi \in \mathfrak{g}^*$ can be defined by $\chi(x) := (x|E)$ for $x \in \mathfrak{g}$. Note that

$$\mathfrak{g}^E = \{z, e, e^2, \dots, e^{n-1} \mid H_0, H_1, \dots, H_{n-1}\}, \quad \dim(\mathfrak{g}^E) = (n|n),$$

where $H_0 = \sum_{i=1}^{n}(-1)^{i-1} f_{i,i}$, $H_1 = \sum_{i=1}^{n-1}(-1)^i f_{i,i+1}$, $\dots$, $H_{n-1} = (-1)^{n-1} f_{1,n}$. Let

$$\mathfrak{m} = \bigoplus_{j=2}^{n} \mathfrak{g}_{2-2j}.$$

Note that $\mathfrak{m}$ is generated by $e_{i+1,i}$ and $f_{i+1,i}$, where $i = 1, \dots, n-1$, and

$$\chi(e_{i+1,i}) = 1, \quad \chi(e_{i+k,i}) = 0 \text{ if } k \geq 2, \quad \chi(f_{i+k,i}) = 0 \text{ if } k \geq 1.$$

The left ideal I_χ and W_χ are defined now as usual.

A. Sergeev defined by induction the elements $e_{i,j}^{(m)}$ and $f_{i,j}^{(m)}$ belonging to $U(\mathfrak{g})$ [S]:

$$e_{i,j}^{(m)} = \textstyle\sum_{k=1}^{n} e_{i,k} e_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} f_{k,j}^{(m-1)},$$

$$f_{i,j}^{(m)} = \textstyle\sum_{k=1}^{n} e_{i,k} f_{k,j}^{(m-1)} + (-1)^{m+1} \sum_{k=1}^{n} f_{i,k} e_{k,j}^{(m-1)}.$$

Then

$$[e_{i,j}, e^{(m)}_{k,l}] = \delta_{j,k} e^{(m)}_{i,l} - \delta_{i,l} e^{(m)}_{k,j}, \quad [e_{i,j}, f^{(m)}_{k,l}] = \delta_{j,k} f^{(m)}_{i,l} - \delta_{i,l} f^{(m)}_{kj},$$
$$[f_{i,j}, e^{(m)}_{k,l}] = (-1)^{m+1}\delta_{j,k} f^{(m)}_{i,l} - \delta_{i,l} f^{(m)}_{k,j}, \quad [f_{i,j}, f^{(m)}_{k,l}] = (-1)^{m+1}\delta_{j,k} e^{(m)}_{i,l} + \delta_{i,l} e^{(m)}_{k,j}.$$

Proposition 18.1. *A. Sergeev [S].*
The elements $\sum_{i=1}^n e^{(2m+1)}_{i,i}$ generate $Z(\mathfrak{g})$.

Theorem 18.1. *[PS2]. $\pi(e^{(m)}_{n,1})$ and $\pi(f^{(m)}_{n,1})$ are Whittaker vectors.*
W_χ has n even generators $\pi(e^{(n+k-1)}_{n,1})$ and n odd generators $\pi(f^{(n+k-1)}_{n,1})$, for $k = 1,\dots,n$. The generators of W_χ can be identified with elements of $\mathfrak{g}^E$:

$$\pi(e^{(n)}_{n,1}) \xrightarrow{P} z, \quad \pi(e^{(n+k-1)}_{n,1}) \xrightarrow{P} e^{k-1}, \quad k = 2,\dots,n,$$

$$\pi(f^{(n+k-1)}_{n,1}) \xrightarrow{P} H_{k-1}, \quad k = 1,\dots,n.$$

Corollary 18.1. *The natural homomorphism $U(\mathfrak{g})^{ad\mathfrak{m}} \longrightarrow W_\chi$ is surjective.*

Let $\mathfrak{p} := \oplus_{j\geq 0}\mathfrak{g}_j$. Let $\mathfrak{f} =< e_{i,i}, f_{i,i} \mid i = 1,\dots,n >$, and let $\vartheta : U(\mathfrak{p}) \longrightarrow U(\mathfrak{f})$ be the Harish-Chandra homomorphism.
Denote

$$x_i = e_{i,i}, \quad \xi_i = (-1)^{i+1} f_{i,i}.$$

Theorem 18.2. *[PS2]. The Harish-Chandra homomorphism is injective.*
Under the Harish-Chandra homomorphism:

$$\vartheta(\pi(e^{(n+k-1)}_{n,1})) = [\textstyle\sum_{i_1\geq i_2\geq\ldots\geq i_k}(x_{i_1} + (-1)^{k+1}\xi_{i_1})\ldots(x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{even},$$
$$\vartheta(\pi(f^{(n+k-1)}_{n,1})) = [\textstyle\sum_{i_1\geq i_2\geq\ldots\geq i_k}(x_{i_1} + (-1)^{k+1}\xi_{i_1})\ldots(x_{i_{k-1}} - \xi_{i_{k-1}})(x_{i_k} + \xi_{i_k})]_{odd}.$$

Theorem 18.3. *[PS2].*

$$\pi(e^{(n+1)}_{n,1}) = \pi(\frac{1}{2}\sum_{i=1}^n e^2_{i,i} + \sum_{i=1}^{n-1} e_{i,i+1} + \sum_{i<j}(-1)^{i-j} f_{i,i} f_{j,j} + \frac{1}{2}z^2 - z).$$

One can define odd generators $\Phi_0,\dots,\Phi_{n-1}$ of W_χ as follows:

$$\begin{aligned}\Phi_0 &= \pi(f_{n,1}^{(n)}) = \pi(H_0),\\ \Phi_1 &= [\pi(e_{n,1}^{(n+1)}), \Phi_0],\\ &\dots\\ \Phi_{n-1} &= [\pi(e_{n,1}^{(n+1)}), \Phi_{n-2}].\end{aligned}$$

Then

$$[\Phi_m, \Phi_p] = 0, \text{ if } m+p \text{ is odd,}$$
$$[\Phi_m, \Phi_p] \in Z(\mathfrak{g}), \text{ if } m+p \text{ is even.}$$

Lemma 18.1. *For odd m and p we have*

$$[\pi(e_{n,1}^{(n+m)}), \pi(e_{n,1}^{(n+p)})] = 0.$$

We set

$$z_i = \pi(e_{n,1}^{(n+i)}) \quad \text{for odd } i,$$

$$z_i = [\Phi_0, \Phi_i] \quad \text{for even } i.$$

Theorem 18.4. *[PS2]. Elements $z_0, \dots, z_{n-1}$ are algebraically independent in W_χ. Together with $\Phi_0, \dots, \Phi_{n-1}$ they form a complete set of generators in W_χ.*

Example 18.1. *$n = 2$, $\mathfrak{g} = Q(2)$.*

Let

$$e = e_{1,2}, \quad h = diag(1,-1), \quad f = e_{2,1}, \quad z = e_{1,1} + e_{2,2},$$
$$E = f_{1,2}, \quad H_0 = f_{1,1} - f_{2,2}, \quad H_1 = -f_{1,2}.$$

Then

$$\mathfrak{g}^E = \{z, e \mid H_0, H_1\}, \quad dim(\mathfrak{g}^E) = (2|2).$$

According to Theorem 7, W_χ has 2 even generators: $\pi(e_{2,1}^{(2)})$, $\pi(e_{2,1}^{(3)})$, and 2 odd generators: $\pi(f_{2,1}^{(2)})$, $\pi(f_{2,1}^{(3)})$. Let

$$\Phi_0 = \pi(f_{2,1}^{(2)}) = \pi(H_0),$$

$$\Phi_1 = [\pi(e_{2,1}^{(3)}), \Phi_0] = 2\pi(f_{2,1}^{(3)}) = 2\pi(-f_{1,2} + f_{1,1}e_{2,2} + f_{2,2}e_{1,1}).$$

Let

$$z_0 = 2\pi(e_{2,1}^{(2)}) = 2\pi(z),$$

$$z_1 = \pi(e_{2,1}^{(3)}) = \pi(e_{1,2} + e_{1,1}^2 + e_{2,2}^2 + e_{1,1}e_{2,2} - e_{1,1} - e_{2,2} - f_{1,1}f_{2,2}).$$

The nonzero commutation relations between the generators of W_χ are as follows:

$$[z_1,\Phi_0]=\Phi_1,\quad [z_1,\Phi_1]=4z_1\Phi_0-2\Phi_1-z_0^2\Phi_0+2z_0\Phi_0,$$

$$[\Phi_0,\Phi_0]=z_0,\quad [\Phi_1,\Phi_1]=4\Phi_1\Phi_0-4z_0z_1+\frac{3}{4}z_0^3-z_0^2.$$

Note that

$$[\Phi_1,\Phi_1]=\frac{1}{3}\pi(-8(e_{1,1}^{(3)}+e_{2,2}^{(3)})+2z^3+4z^2).$$

Hence by Proposition 5

$$[\Phi_1,\Phi_1]\in\pi(Z(\mathfrak{g})).$$

Remark 18.2. *L. Zhao studied finite W-algebras for $\mathfrak{g}=Q(n)$ [Z]. He proved that the definition of the finite W-algebra is independent of the choices of the isotropic subspaces $\mathfrak{l}$ and the good $\mathbb{Z}$-gradings. He also established a Skryabin type equivalence between the category of W_χ-modules and a category of certain $\mathfrak{g}$-modules.*

Conjecture 18.1. *[PS2]. Irreducible modules over the principal finite W-algebra for $Q(n)$ are finite-dimensional.*

Conjecture 18.2. *[PS2]. In the case when $\dim(\mathfrak{g}_{-1})_{\bar{1}}$ is even, it is possible to find a set of generators of the principal finite W-algebra for $\mathfrak{g}$ such that even generators commute, and the commutators of odd generators are in $\pi(Z(\mathfrak{g}))$.*

19. The super-Yangian of $\mathbf{Q}(\mathbf{n})$

In this section, we describe the principal finite W-algebra for $Q(n)$ as a factor algebra of the super-Yangian of $Q(1)$.

The super-Yangian $Y(Q(n))$ was studied by M. Nazarov and A. Sergeev [N, NS]. Note that $Y(Q(n))$ is the associative unital superalgebra over $\mathbb{C}$ with the countable set of generators

$$T_{i,j}^{(m)},\ \text{where } m=1,2,\ldots \text{ and } i,j=\pm1,\pm2,\ldots,\pm n.$$

The $\mathbb{Z}_2$-grading of the algebra $Y(Q(n))$ is defined as follows:

$$p(T_{i,j}^{(m)})=p(i)+p(j),\ \text{where } p(i)=0 \text{ if } i>0 \text{ and } p(i)=1 \text{ if } i<0.$$

To describe defining relations for the generators of $Y(Q(n))$ we employ the formal series in $Y(Q(n))[[u^{-1}]]$:

$$T_{i,j}(u)=\delta_{ij}\cdot 1+T_{i,j}^{(1)}u^{-1}+T_{i,j}^{(2)}u^{-2}+\ldots.$$

For all indices i, j, k, l we have the relations

$$(u^2 - v^2)[T_{i,j}(u), T_{k,l}(v)] \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)}$$
$$= (u+v)(T_{k,j}(u)T_{i,l}(v) - T_{k,j}(v)T_{i,l}(u))$$
$$-(u-v)(T_{-k,j}(u)T_{-i,l}(v) - T_{k,-j}(v)T_{i,-l}(u)) \cdot (-1)^{p(k)+p(l)}, \quad (3)$$

where v is a formal parameter independent of u, so that (3) is an equality in the algebra of formal Laurent series in u^{-1}, v^{-1} with coefficients in $Y(Q(n))$. We also have the relations

$$T_{i,j}(-u) = T_{-i,-j}(u). \quad (4)$$

Relations (3) and (4) are equivalent to the following defining relations:

$$([T_{i,j}^{(m+1)}, T_{k,l}^{(r-1)}] - [T_{i,j}^{(m-1)}, T_{k,l}^{(r+1)}]) \cdot (-1)^{p(i)p(k)+p(i)p(l)+p(k)p(l)} =$$
$$T_{k,j}^{(m)}T_{i,l}^{(r-1)} + T_{k,j}^{(m-1)}T_{i,l}^{(r)} - T_{k,j}^{(r-1)}T_{i,l}^{(m)} - T_{k,j}^{(r)}T_{i,l}^{(m-1)}$$
$$+(-1)^{p(k)+p(l)}(-T_{-k,j}^{(m)}T_{-i,l}^{(r-1)} + T_{-k,j}^{(m-1)}T_{-i,l}^{(r)} + T_{k,-j}^{(r-1)}T_{i,-l}^{(m)} - T_{k,-j}^{(r)}T_{i,-l}^{(m-1)}), \quad (3')$$
$$T_{-i,-j}^{(m)} = (-1)^m T_{i,j}^{(m)}, \quad (4')$$

where $m, r = 1, \ldots$ and $T_{ij}^{(0)} = \delta_{ij}$.

Theorem 19.1. *[PS2]. There exists a surjective homomorphism:*

$$\varphi : Y(Q(1)) \longrightarrow W_\chi,$$

defined as follows:

$$\varphi(T_{1,1}^{(k)}) = (-1)^k \pi(e_{n,1}^{(n+k-1)}), \quad \varphi(T_{-1,1}^{(k)}) = (-1)^k \pi(f_{n,1}^{(n+k-1)}) \text{ for } k = 1, 2, \ldots.$$

20. The case of $\mathfrak{g} = \Gamma(\sigma_1, \sigma_2, \sigma_3)$

In this section, we describe the principal finite W-algebra for the exceptional Lie superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ in terms of generators and relations. We follow the construction of this Lie superalgebra given by M. Scheunert [Sch].

Let $\sigma_1, \sigma_2, \sigma_3$ be complex numbers such that $\sigma_1 + \sigma_2 + \sigma_3 = 0$. The family of Lie superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ is defined as follows:

$$\Gamma(\sigma_1, \sigma_2, \sigma_3) = \Gamma_{\bar{0}} \oplus \Gamma_{\bar{1}}, \text{ where}$$

$$\Gamma_{\bar{0}} = \mathfrak{sl}(2)_{\bar{1}} \oplus \mathfrak{sl}(2)_{\bar{2}} \oplus \mathfrak{sl}(2)_{\bar{3}}, \quad \Gamma_{\bar{1}} = V_1 \otimes V_2 \otimes V_3,$$

where V_i is the standard $\mathfrak{sl}(2)_i$-module for $i = 1, 2, 3$.

Let $\mathfrak{sl}(2)_i =< X_i, H_i, Y_i >$, where $\{X_i, H_i, Y_i\}$ is the standard basis in each $\mathfrak{sl}(2)_i$ for $i = 1, 2, 3$. Let $V_i =< e_i, f_i >$ and let $P_i : V_i \times V_i \to \mathfrak{sl}(2)_i$ be the $\mathfrak{sl}(2)_i$-invariant bilinear mapping given by

$$P_i(e_i, e_i) = 2X_i, \quad P_i(f_i, f_i) = -2Y_i, \quad P_i(e_i, f_i) = P_i(f_i, e_i) = -H_i.$$

Let ψ_i be a non-degenerate skew-symmetric form on V_i:

$$\psi_i(e_i, f_i) = -\psi_i(f_i, e_i) = 1.$$

Note that $[\Gamma_{\bar{0}}, \Gamma_{\bar{1}}]$ is the tensor product of the standard representations of $\mathfrak{sl}(2)_i$ in V_i, and $[\Gamma_{\bar{1}}, \Gamma_{\bar{1}}]$ is given by the formula

$$[x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes y_3] = \sigma_1 \psi_2(x_2, y_2)\psi_3(x_3, y_3)P_1(x_1, y_1) +$$
$$\sigma_2 \psi_1(x_1, y_1)\psi_3(x_3, y_3)P_2(x_2, y_2) + \sigma_3 \psi_1(x_1, y_1)\psi_2(x_2, y_2)P_3(x_3, y_3),$$

where $x_i, y_i \in V_i$, $i = 1, 2, 3$.

Remark 20.1. *The superalgebra $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ is simple if and only if $\sigma_i \neq 0$ for $i = 1, 2, 3$. $\Gamma(\sigma_1, \sigma_2, \sigma_3) \cong \Gamma(\sigma'_1, \sigma'_2, \sigma'_3)$ if and only if the sets $\{\sigma'_i\}$ and $\{\sigma_i\}$ are obtained from each other by a permutation and multiplication of all elements of one set by a nonzero complex number (see [Sch]). Thus $\Gamma(\sigma_1, \sigma_2, \sigma_3)$ is a one-parameter family of deformations of $osp(4|2)$. Note that $\Gamma(1, -1-\alpha, \alpha) \cong D(2, 1; \alpha)$, where $\alpha \neq 0, -1$ (see [K]).*

We consider the non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ given as follows:

$$(X_i, Y_i) = \tfrac{1}{\sigma_i}, \quad (H_i, H_i) = \tfrac{2}{\sigma_i},$$
$$(e_1 \otimes e_2 \otimes e_3, f_1 \otimes f_2 \otimes f_3) = -2, \quad (e_1 \otimes e_2 \otimes f_3, f_1 \otimes f_2 \otimes e_3) = 2,$$
$$(e_1 \otimes f_2 \otimes e_3, f_1 \otimes e_2 \otimes f_3) = 2, \quad (f_1 \otimes e_2 \otimes e_3, e_1 \otimes f_2 \otimes f_3) = 2.$$

Let $\mathfrak{sl}(2) =< e, h, f >$, where

$$e = X_1 + X_2 + X_3, \quad h = H_1 + H_2 + H_3, \quad f = Y_1 + Y_2 + Y_3.$$

Then e is a regular nilpotent element, h defines a Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \oplus_{j=-3}^{3} \mathfrak{g}_j, \text{ where}$$

$$\mathfrak{g}_3 =< e_1 \otimes e_2 \otimes e_3 >, \quad \mathfrak{g}_2 =< X_1, X_2, X_3 >,$$
$$\mathfrak{g}_1 =< e_1 \otimes e_2 \otimes f_3, e_1 \otimes f_2 \otimes e_3, f_1 \otimes e_2 \otimes e_3 >,$$
$$\mathfrak{g}_0 =< H_1, H_2, H_3 >, \quad \mathfrak{g}_{-1} =< e_1 \otimes f_2 \otimes f_3, f_1 \otimes e_2 \otimes f_3, e_1 \otimes e_2 \otimes f_3 >,$$
$$\mathfrak{g}_{-2} =< Y_1, Y_2, Y_3 >, \quad \mathfrak{g}_{-3} =< f_1 \otimes f_2 \otimes f_3 > .$$

Note that $\dim(\mathfrak{g}^e) = (3|3)$. Explicitly,

$(\mathfrak{g}^e)_{\bar{0}} =< X_1, X_2, X_3 >$,

$(\mathfrak{g}^e)_{\bar{1}} =< e_1 \otimes f_2 \otimes e_3 - e_1 \otimes e_2 \otimes f_3, f_1 \otimes e_2 \otimes e_3 - e_1 \otimes e_2 \otimes f_3, e_1 \otimes e_2 \otimes e_3 >$.

In $\mathfrak{g}$ we consider a nilpotent subalgebra $\mathfrak{m}=\mathfrak{g}_{-3}\oplus\mathfrak{g}_{-2}\oplus\mathfrak{l}$, where $\mathfrak{l}$ is a one-dimensional Lagrangian subspace of $\mathfrak{g}_{-1}$: $\mathfrak{l}=<e_1\otimes f_2\otimes f_3>$. Note that $\mathfrak{m}$ is generated by Y_1,Y_2,Y_3 and $e_1\otimes f_2\otimes f_3$. We have that

$$\chi(Y_i)=\frac{1}{\sigma_i},\quad \chi(e_1\otimes f_2\otimes f_3)=0.$$

Let $\theta=f_1\otimes e_2\otimes f_3-f_1\otimes f_2\otimes e_3$. Then $\theta\in\mathfrak{g}_{-1}\cap\mathfrak{l}^{\perp}$, $\pi(\theta)\in W_\chi$, $\quad \pi(\theta)^2=-2$.

Even generators of W_χ are:

$C_1=\pi(2X_1+\sigma_1(\frac{1}{2}H_1^2-H_1))$,
$C_2=\pi(2X_2+\frac{1}{2}\sigma_2H_2^2+(f_1\otimes e_2\otimes f_3)(e_1\otimes e_2\otimes f_3))$,
$C_3=\pi(2X_3+\frac{1}{2}\sigma_3H_3^2+(f_1\otimes f_2\otimes e_3)(e_1\otimes f_2\otimes e_3))$.

Odd generators of W_χ are

$R_1=\pi(2(e_1\otimes f_2\otimes e_3-e_1\otimes e_2\otimes f_3)+\sigma_1H_1(f_1\otimes e_2\otimes f_3-f_1\otimes f_2\otimes e_3))$,
$R_2=\pi(2(f_1\otimes e_2\otimes e_3-e_1\otimes e_2\otimes f_3)$
$\quad +(\sigma_1H_1-\sigma_3H_3)(f_1\otimes e_2\otimes f_3)-\sigma_2H_2(f_1\otimes f_2\otimes e_3))$,
$R_3=\pi(4(e_1\otimes e_2\otimes e_3)-\sigma_1H_1R_2-4\sigma_1(f_1\otimes e_2\otimes f_3)X_1$
$\quad -2(\sigma_1H_1(e_1\otimes e_2\otimes f_3)+\sigma_2H_2(e_1\otimes f_2\otimes e_3)+\sigma_3H_3(e_1\otimes e_2\otimes f_3)))$,

and $\pi(\theta)$. Note that the generators C_i and R_i for $i=1,2,3$ can be identified with the corresponding elements of $\mathfrak{g}^e$ under the mapping P (see Remark 8). The quadratic Casimir element of $\mathfrak{g}$ is

$$\Omega=\Sigma_{i=1}^3(\frac{\sigma_i}{2}H_i^2+2X_i)-(e_1\otimes e_2\otimes f_3)(f_1\otimes f_2\otimes e_3)-(e_1\otimes f_2\otimes e_3)(f_1\otimes e_2\otimes f_3).$$

Hence

$$\pi(\Omega)=C_1+C_2+C_3-\frac{1}{2}R_1\pi(\theta).$$

Theorem 20.1. *[P]. The principal finite W-algebra W_χ is generated by even elements $\pi(\Omega)$, C_1 and C_2, and odd element $\pi(\theta)$. The relations are*

$$[C_1,C_2]=0,\quad [\pi(\theta),C_i]=R_i\mp\tfrac{\sigma_i}{2}\pi(\theta),\quad i=1,2,$$
$$[C_2,R_1]=-\tfrac{\sigma_2}{2}R_1+R_3,\quad [C_1,R_2]=\tfrac{\sigma_1}{2}R_2+R_3,$$
$$[R_i,R_i]=8\sigma_iC_i-2\sigma_iR_i\pi(\theta),\quad i=1,2,$$
$$[R_1,R_2]=-4(\sigma_1C_2+\sigma_2C_1+\sigma_3\pi(\Omega))+(\sigma_1R_2+\sigma_2R_1)\pi(\theta),$$
$$[R_i,\pi(\theta)]=\mp2\sigma_i,\quad i=1,2,\quad [\pi(\theta),\pi(\theta)]=-4,$$
$$[\pi(\Omega),\pi(\theta)]=0,\quad [\pi(\Omega),C_i]=0,\ i=1,2,\quad [\pi(\Omega),R_i]=0,\ i=1,2,3.$$

21. The case of $\mathfrak{g} = \mathfrak{osp}(\mathbf{1}|\mathbf{2})$

In this section, we describe the principal finite W-algebra for $\mathfrak{osp}(1|2)$.

Form: $(a|b) = \frac{1}{2}\mathrm{str}(ab)$
$\mathfrak{g} =< e, f, h \mid \theta, r >$, where

$$e = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right), f = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right), h = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right),$$

$$\theta = \left(\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right), r = \left(\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Note that e is a regular nilpotent element, and h defines a $\mathbb{Z}$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where

$$\mathfrak{g}_{-2} =< f >, \quad \mathfrak{g}_{-1} =< \theta >, \quad \mathfrak{g}_0 =< h >, \quad \mathfrak{g}_1 =< r >, \quad \mathfrak{g}_2 =< e > .$$

$$\mathfrak{m} = \mathfrak{g}_{-2} =< f >, \quad \chi(f) = -\frac{1}{2}.$$

$$\mathfrak{g}^e =< e|r >, \quad \dim \mathfrak{g}^e = (1|1).$$

Note that $\pi(\theta) \in W_\chi$, and $\pi(\theta)^2 = \frac{1}{2}$.
Even generator of W_χ is $\pi(\Omega)$, where $\Omega = 2e + h - h^2 + 2r\theta$ is the Casimir element of $\mathfrak{g}$.
Odd generators of W_χ are $R = \pi(r - h\theta)$ and $\pi(\theta)$.
Note that generators $\pi(\Omega)$ and R are identified with elements of $\mathfrak{g}^e$:

$$\tfrac{1}{2}\pi(\Omega) \xrightarrow{P} e,$$

$$R \xrightarrow{P} r.$$

Theorem 21.1. *The principal finite W-algebra W_χ is generated by $\pi(\Omega)$ and two odd generators: $\pi(\theta)$ and R. The defining relations are*

$$[\pi(\Omega), R] = [\pi(\Omega), \pi(\theta)] = 0,$$
$$[R, R] = \pi(\Omega), \quad [R, \pi(\theta)] = -\tfrac{1}{2}, \quad [\pi(\theta), \pi(\theta)] = 1.$$

22. The case of $\mathfrak{g} = \mathfrak{osp}(1|2n)$

In this section, we present partial results for the principal finite W-algebra for $\mathfrak{osp}(1|2n)$, where $n \geq 2$, and make a conjecture for this case.

Form: $(a|b) = -\mathrm{str}(ab)$.

We will use the following notations for some elementary matrices in $\mathfrak{osp}(1|2n)$:

$$\left(\begin{array}{c|cccc|cccc}
0 & s_1 & s_2 & \dots & \mathbf{s_n} & r_1 & r_2 & \dots & r_n \\ \hline
r_1 & h_1 & x_1 & \dots & \dots & p_1 & \dots & \dots & \dots \\
\dots & y_1 & h_2 & x_2 & \dots & .. & p_2 & \dots & \dots \\
\dots & \dots & \dots & \dots & x_{n-1} & \dots & \dots & \dots & \dots \\
r_n & \dots & \dots & y_{n-1} & h_n & \dots & \dots & \dots & p_n \\ \hline
s_1 & q_1 & \dots & \dots & \dots & h_1 & y_1 & 0 & 0 \\
\dots & \dots & q_2 & \dots & \dots & x_1 & h_2 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & y_{n-1} \\
\mathbf{s_n} & \dots & \dots & \dots & q_n & 0 & 0 & x_{n-1} & h_n
\end{array}\right).$$

Let $\mathfrak{sl}(2) = < e, h, f >$, where $e = (x_1 + \dots x_{n-1}) + p_n$,
$h = \mathrm{diag}(0|2n-1, 2n-3, \dots, 3, 1; -2n+1, -2n+3, \dots, -3, -1)$,
$f = (\sum_{k=1}^{n-1} k(2n-k)y_k) + n^2 q_n$.
Note that e is a regular nilpotent element, and h defines a Dynkin $\mathbb{Z}$-grading of $\mathfrak{g}$ whose degrees on the elementary matrices are

$$\left(\begin{array}{c|cccc|cccc}
0 & -2n+1 & \dots & -3 & -\mathbf{1} & 2n-1 & \dots & 3 & 1 \\ \hline
2n-1 & 0 & 2 & \dots & \dots & 4n-2 & \dots & 2n+2 & 2n \\
\dots & -2 & 0 & 2 & \dots & 4n-4 & \dots & 2n & 2n-2 \\
3 & \dots & \dots & 0 & 2 & \dots & \dots & 6 & 4 \\
1 & \dots & -4 & -2 & 0 & 2n & \dots & 4 & 2 \\ \hline
-2n+1 & -4n+2 & \dots & -2n-2 & -2n & 0 & -2 & 0 & 0 \\
\dots & -4n+4 & \dots & -2n & -2n+2 & 2 & 0 & -2 & 0 \\
-3 & \dots & \dots & -6 & -4 & \dots & \dots & 0 & -2 \\
-\mathbf{1} & -2n & \dots & -4 & -2 & \dots & \dots & 2 & 0
\end{array}\right).$$

Note that $\dim \mathfrak{g}^e = (n|1)$, $(\mathfrak{g}^e)_{\bar{1}} = < r_1 >$, $\mathfrak{g}_{-1} = < \theta >$, where $\theta = s_n$, $\dim \mathfrak{g}_{-1} = 1$.

Note that $\mathfrak{m} = \oplus_{j \leq -2} \mathfrak{g}_j$, and $\mathfrak{g}_{-2} = < y_i, q_n >$, $i = 1, \dots, n-1$.

$\mathfrak{m}$ is generated by y_i, q_n, $\chi(y_i) = 2$, for $i = 1, \dots, n-1$, and $\chi(q_n) = 1$.

$\pi(\theta) \in W_\chi$, $\pi(\theta)^2 = -1$.

Conjecture 22.1. *[P]. The principal finite W-algebra W_χ is generated by the first n Casimir elements in $\pi(Z(\mathfrak{g}))$ and odd elements $\pi(\theta)$ and R, where R is induced* *by r_1 so that*

$$[R,R] \in \pi(Z(\mathfrak{g})), \quad [R,\pi(\theta)] \in \pi(Z(\mathfrak{g})), \quad [\pi(\theta),\pi(\theta)] = -2.$$

Acknowledgments

The author would like to thank V. Serganova for very helpful discussions.

23. References

[A] Eiichi Abe, *Hopf Algebras*, Cambridge Tracts in Mathematics **74**, Cambridge University Press (2004).

[B] J. Balog, L. Fehér and L. O'Raifeartaigh, *Toda theory and W-algebra from a gauged WZNW point of view*, Ann. Phys. **203** (1990) 76–136.

[BR] C. Briot, E. Ragoucy, *W-superalgebras as truncations of super-Yangians*, J. Phys. A **36** (2003), no. 4, 1057–1081.

[BG] J. Brundan and S. Goodwin, *Good grading polytopes.* Proc. London Math. Soc. **94** (2007) 155–180.

[BBG] J. Brown, J. Brundan, S. Goodwin, *Principal W-algebras for* $GL(m|n)$, arXiv:1205.0992.

[BK1] J. Brundan and A. Kleshchev *Shifted Yangians and finite W-algebras*, Adv. Math. **200** (2006) 136–195.

[BK2] J. Brundan and A. Kleshchev *Representations of shifted Yangians and finite W-algebras*, Mem. Amer. Math. Soc. **196** (2008), no. 918, 107 pp.

[C] R. Carter, *Finite groups of Lie type: Conjugacy classes and complex characters.* Pure and Applied Math., John Wiley & Sons, Inc., New York (1985).

[DK] A. De Sole and V. Kac, *Finite vs affine W-algebras*, Jpn. J. Math. **1** (2006) 137–261.

[EK] A. G. Elashvili and V. G. Kac, *Classification of good gradings of simple Lie algebras*, Lie groups and invariant theory (E. B. Vinberg ed.), Amer. Math. Soc. Transl. **213** (2005) 85–104.

[F1] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, *Generalized Toda theories and W-algebras associated with integral gradings*, Ann. Phys. **213** (1992) 1–20.

[F2] L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, *On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories*, Phys. Rep. **222** (1992) 1–64.

[GG] W. L. Gan and V. Ginzburg, *Quantization of Slodowy slices*. Internat. Math. Res. Notices **5** (2002) 243-255.

[H] C.Hoyt, *Good gradings of basic Lie superalgebras*, Israel J. Math. **192** (2012) 251–280.

[K] V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977) 8–96.

[KW] V. G. Kac and M. Wakimoto, *Integrable highest weight modules over affine superalgebras and number theory*. Lie theory and geometry, 415–456, Progr. Math., **123**, Birkhäuser Boston, Boston, MA, 1994.

[Ko] B. Kostant, *On Wittaker vectors and representation theory*, Invent. Math. **48** (1978) 101–184.

[L1] I. Losev, *Finite W-algebras*, Proceedings of the International Congress of Mathematicians. Volume III, 1281–1307, Hindustan Book Agency, New Delhi, 2010. arXiv:1003.5811v1.

[L2] I. Losev, *Quantized symplectic actions and W-algebras*, J. Amer. Math. Soc. **23** (2010) 35–59.

[L3] I. Losev, *Finite-dimensional representations of W-algebras*, Duke Math. J. **159** (2011), no.1, 99–143.

[M] A. Molev, *Yangians and classical Lie algebras*, Mathematical Surveys and Monographs, **143**, Amer. Math. Soc., Providence, RI, (2007).

[N] M. Nazarov, *Yangian of the Queer Lie superalgebra*, Commun. Math. Phys. **208** (1999) 195–223.

[NS] M. Nazarov, A. Sergeev, *Centralizer construction of the Yangian of the queer Lie superalgebra*, Studies in Lie Theory, 417–441, Progr. Math., **243** (2006).

[P] E. Poletaeva, *On Kostant's Theorem for Lie superalgebras*. In: Springer INdAM Series: Advances in Lie Superalgebras, P. Papi, M. Gorelik (eds). To be published.

[PS1] E. Poletaeva, V. Serganova, *On finite W-algebras for Lie superalgebras in the regular case*, In: "Lie Theory and Its Applications in Physics", V. Dobrev (ed), IX International Workshop. 20-26 June 2011, Varna, Bulgaria. Springer Proceedings in Mathematics and Statistics, Vol. **36** (2013) 487–497.

[PS2] E. Poletaeva, V. Serganova, *On Kostant's theorem for the Lie superalgebra $Q(n)$*, in preparation.

[Pr1] A. Premet, *Special transverse slices and their enveloping algebras*, Adv. Math. **170** (2002) 1–55.

[Pr2] A. Premet, *Enveloping algebras of Slodowy slices and the Joseph ideal*, J. Eur. Math. Soc. **9** (2007) 487–543.

[Pr3] A. Premet, *Primitive ideals, non-restricted representations and finite W-algebras*, Mosc. Math. J. **7** (2007) 743–762.

[S] A. Sergeev, *The centre of envelloping algebra for Lie superalgebra $Q(n,\mathbb{C})$*, Letters in Math. Phys. 7 (1983) 177–179.

[Sch] M. Scheunert, *The Theory of Lie Superalgebras*. In: Lecture Notes in Mathematics, **716**, Springer, Berlin (1979).

[W] W. Wang, *Nilpotent orbits and finite W-algebras*, Geometric representation theory and extended affine Lie algebras, Fields Inst. Commun. **59**, Amer. Math. Soc., Providence, RI, (2011), 71–105; arXiv:0912.0689v2.

[Z] L. Zhao, *Finite W-superalgebras for queer Lie superalgebras*. arXiv:1012.2326v2.